THEORY OF PLATES AND SHELLS IN THE REFERENCE STATE

By

LUI MORIS HABIP

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TO MY PARENTS

PREFACE

The results reported in this dissertation were partially obtained by December, 1963 and presented by the author in a seminar at the University of Florida on January 15, 1964. Several portions* of the MS have been submitted for publication and are currently being reviewed.

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*L. M. Habip, "A note on the equations of motion of plates in the reference state."

L. M. Habip, "On the theory of plates in the reference state."

L. M. Habip and I. K. Ebcioglu, "On the equations of motion of shells in the reference state."

L. M. Habip, "On the theory of shells in the reference state."

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KEY TO SYMBOLS

All symbols are explained in the text when first introduced. In addition, the list below has been compiled for symbols that appear frequently. Latin indices denote space functions, while Greek indices are reserved for subtensors and surface functions. The former take the values 1, 2, 3 unless stated otherwise, and the latter take the values 1, 2 only. Repeated indices are to be summed over their respective range. Vector quantities are indicated by subtildes. A dot between two such quantities denotes dot product. Tildes indicate prescribed quantities, bars, "shifted" quantities. All coordinate systems are right handed. The stress tensor components satisfy the usual sign convention.

θ^i	right handed convected general curvilinear coordinate system; convected normal coordinate system for plate or shell
τ	time
Θ	temperature
g_{ij}	metric tensor of the undeformed body
g	determinant of the components g_{ij}
$(\ )\vert_i$	covariant differentiation with respect to θ^i and g_{ij}
ρ_o	density of the undeformed body
$\underset{\sim}{g}_i, \underset{\sim}{g}^i$	covariant and contravariant unit base vectors of the undeformed body
$\underset{\sim}{G}_i, \underset{\sim}{G}^i$	covariant and contravariant base vectors of the deformed body

v_i , v^i	covariant and contravariant components of the displacement vector referred to $\underset{\sim}{g}^i$ and $\underset{\sim}{g}_i$, respectively
γ_{ij}	covariant strain tensor
${}_{\circ}\underset{\sim}{t}$	stress vector per unit area of the undeformed body
s^{ij}	contravariant stress tensor measured per unit area of the undeformed body, when ${}_{\circ}\underset{\sim}{t}$ is referred to $\underset{\sim}{G}_i$
t^{ij}	contravariant stress tensor measured per unit area of the undeformed body when ${}_{\circ}\underset{\sim}{t}$ is referred to $\underset{\sim}{g}_i$
${}_{\circ}\underset{\sim}{n}$	unit normal to the undeformed position of a surface in the deformed body associated with ${}_{\circ}\underset{\sim}{t}$
${}_{\circ}n_i$	covariant components of ${}_{\circ}\underset{\sim}{n}$ referred to $\underset{\sim}{g}^i$
${}_{\circ}F^i$	contravariant components of body force vector per unit mass of undeformed body referred to $\underset{\sim}{g}_i$
${}_{\circ}f^i$	contravariant components of acceleration vector referred to $\underset{\sim}{g}_i$
Σ^*	strain energy function per unit volume of the undeformed body
C^{ijrs}	isothermal stiffnesses
α_{ij}	strain-temperature coefficients at constant stress
δ^i_j	Kronecker symbol
ϵ_{ijk} , $\bar{\epsilon}_{\alpha\beta}$	ϵ -systems defined in (20) and (50), respectively
e_{ijk}	permutation symbol
δ	variation
${}_{\circ}V$, dV	volume of the undeformed body, related element of volume

${}_{0}A_s$, ${}_{0}A_v$, dA	area of the bounding surface of ${}_{0}V$ where the stress and displacement vectors are prescribed, respectively; related element of area
${}_{0}a$	middle plane of the undeformed plate; middle surface of the undeformed shell
a_{ij}	metric tensor associated with ${}_{0}a$
a	determinant of the components $a_{\alpha\beta}$
$(\ )\Vert_{\alpha}$	covariant differentiation with respect to θ^i and a_{ij}
$b_{\alpha\beta}$, b, $c_{\alpha\beta}$	second fundamental form of ${}_{0}a$, determinant of the components $b_{\alpha\beta}$, third fundamental form of ${}_{0}a$
μ^{α}_{β}	expression defined in (47)
μ	determinant of the components μ^{α}_{β}
$(\mu^{-1})^{\alpha}_{\beta}$	"shell-tensor," inverse of μ^{α}_{β}
$2h$	uniform thickness of plate or shell
${}_{0}\mathcal{C}$, ${}_{0}\mathcal{C}_s$, ${}_{0}\mathcal{C}_v$, ds	intersection of ${}_{0}a$ and edge boundary of plate or shell; parts where the stress and displacement vectors are prescribed, respectively; related element of arc length
u_{α}, ψ_{α}, w, w_1	displacement functions introduced in (38) for plates, in (51) for shells
k^{α}_{β}	expression defined in (40)
${}_{0}\gamma_{\alpha\beta}$, ${}_{1}\gamma_{\alpha\beta}$, ${}_{2}\gamma_{\alpha\beta}$, ${}_{0}\gamma_{\alpha 3}$, ${}_{1}\gamma_{\alpha 3}$, ${}_{0}\gamma_{33}$	expressions introduced in (39) and given, in terms of displacement functions, in (89) for plates, in (104) for shells
s^{i}_{α}, s^{α}, t^{α}, s, t	components of ${}_{0}\underset{\sim}{t}$ referred to $\underset{\sim}{g}_i$, expressions defined in (87) for plates, in (101) for shells

$N^{\alpha\beta}, M^{\alpha\beta}, K^{\alpha\beta}$; Q^{α}, T^{α}, N^{33}	stress and couple resultants, defined in (40) for plates, in (57) for shells
$'N^{\alpha\beta}, 'M^{\alpha\beta}, 'K^{\alpha\beta}$	stress and couple resultants for shells, defined in (57)
$\mathcal{F}^{\alpha}, \mathcal{F}, \mathfrak{M}^{\alpha}, \mathfrak{M}$	body force and couple resultants, defined in (40) for plates, in (57) for shells
$f^{\alpha}, f, c^{\alpha}, c$	acceleration resultants, defined in (40) for plates, in (57) for shells
$p^{\alpha}, p, m^{\alpha}, m$	effective external loads, defined in (40) for plates, in (57) for shells
Σ	strain energy function per unit area of ${}_{\bullet}a$, defined in (41) for plates, in (59) for shells
${}_nB^{\alpha\beta\delta\nu}, {}_nB^{\alpha\beta 33}, {}_nB^{\alpha 3\beta 3}, B^{3333}$	expressions defined in (91) for plates, in (106) for shells
${}_n\Theta^{\alpha\beta}, {}_n\Theta^{\alpha 3}, \Theta^{33}$	thermal stress and couple resultants, defined in (92) for plates, in (107) for shells

Abstract of Dissertation Presented to the Graduate Council in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

THEORY OF PLATES AND SHELLS IN THE REFERENCE STATE

By

Lui Moris Habip

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Chairman: Dr. Ibrahim K. Ebcioglu
Major Department: Engineering Science and Mechanics

The fundamental equations of the theory of plates and shells in terms of a reference state have been obtained on the basis of the three-dimensional theory of elasticity and by following the alternate methods of integrating the stress equations of motion across the thicknesss of the undeformed thin body, and that of using a general variational principle, respectively, the latter method leading to a complete set of plate and shell equations.

The theory involves the use of the notion of stress measured per unit area of the undeformed body in contradistinction from the conventional representation of stress which is measured per unit area of the deformed body.

The two methods have been illustrated for the case when the displacement components, when "shifted" for the case of shells, can be assumed to vary linearly across the thickness of the thin body.

The results include strain-displacement relations, equations of motion and mixed boundary conditions for both plates and shells. In addition, general stress-strain relations for plates or shells in terms of a strain energy function defined per unit area of undeformed middle plane

or surface, as well as more specific linear stress-strain relations have been employed for an anisotropic material having one plane of elastic symmetry, and including the effect of a prescribed steady temperature field.

This, then, is a general treatment of the theory of plates and shells in terms of a reference state and contains several "nonlinear" theories of plates and shells available in the literature, as special cases. Eventually, the second method adopted here has the advantage of providing a complete set of fundamental equations consistent with the various linearizations in the strain-displacement relations introduced in these latter theories and which, in view of our results, can now systematically be reappraised.

I. INTRODUCTION

The purpose of this dissertation is to develop the fundamental equations of the theory of plates and shells in terms of a reference state on the basis of the three-dimensional theory of elasticity, and by following the alternate methods of integrating the stress equations of motion across the thickness of the undeformed thin body, and that of using a general variational principle, respectively, the latter leading to a complete set of plate and shell equations.

The theory of the reference state involves the use of the notion of stress measured per unit area of the undeformed body, taken as reference, in contradistinction from the conventional representation of stress which is measured per unit area of the deformed body. The distinction is of significance for arbitrary and yet admissible deformations. This approach illuminates several of the "nonlinear," "finite" or "large deflection" theories currently available in the literature.

The first method which, for the classical theory of plates, dates back to the last century--and has been employed in recent years in developing extensions of it that include transverse shear strain and rotatory inertia effects--is now known to yield, in all generality, the equations of motion of shells, when the conventional measure of stress is used. As to the variational procedure adopted here, several applications of it, within the conventional theories of plates and shells, exist in the literature. Our work provides a general treatment of the

theory of plates and shells in terms of a reference state, and contains the very few earlier attempts discussed in the text.

Most of the notation and preliminary definitions as well as intermediate results and all of the three-dimensional theory that constitutes the basis of the later developments in the text are given in this section. The sections following can be read independently from each other.

The method of integrating the stress equations of motion across the thickness of the undeformed plate and shell are illustrated in Sections II and III, respectively. The variational principle established in Section I is applied to plates in Section IV, and to shells in Section V. Some concluding remarks are given in Section VI. References are given at the end in the order in which they first appear in the text as shown by numbers in square brackets.

I.1. Three-dimensional Theory

In order to render the present work self-contained, we quote from [1] a number of considerations that are necessary for our purpose.

Let a point of a body in the reference state initially at x^i referred to a fixed rectangular cartesian coordinate system move, after the deformation of the body, to y^i in the same coordinate system. We define a general curvilinear coordinate system so that

$$x^i = x^i(\theta^1, \theta^2, \theta^3) , \tag{1}$$

where $x^i(\theta^1, \theta^2, \theta^3)$ is a single-valued function posessing a unique inverse and continuous derivatives up to any required order, except possibly at singular points, lines, or surfaces. It follows that the

Jacobian of (1)

$$\left|\frac{\partial x^i}{\partial \theta^r}\right| \neq 0 \quad , \tag{2}$$

and we assume that

$$\left|\frac{\partial x^i}{\partial \theta^r}\right| > 0 \tag{3}$$

everywhere. The deformation of the body is then defined by

$$y^i = y^i(x^1, x^2, x^3; \tau) \quad , \tag{4}$$

or by

$$y^i = y^i(\theta^1, \theta^2, \theta^3; \tau) \quad , \tag{5}$$

where τ denotes time, the functional forms in (4) and (5) being different in general. The coordinates θ^i are called convected coordinates. The functions $y^i(\theta^1, \theta^2, \theta^3; \tau)$ are assumed to be single-valued and continuously differentiable with respect to θ^i and τ as many times as required, except possibly at singular points, lines, and surfaces, and for each value of τ they have unique inverses. Also, if the deformation is to be possible in a real material, we have that the Jacobian of (5)

$$\left|\frac{\partial y^i}{\partial \theta^r}\right| > 0 \ . \tag{6}$$

The position vector of the point initially at x^i is

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^1, \theta^2, \theta^3) \ . \tag{7}$$

The corresponding position vector of the point y^i is

$$\underset{\sim}{R} = \underset{\sim}{R}(\theta^1, \theta^2, \theta^3; \tau) \ . \tag{8}$$

Base vectors and symmetric metric tensors for the coordinate system θ^i may be defined in both the undeformed and deformed bodies. Thus

unit or base vector

$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial \theta^i} \quad , \quad \underset{\sim}{G}_i = \frac{\partial \underset{\sim}{R}}{\partial \theta^i} \quad ,$$

$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j = \frac{\partial x^r}{\partial \theta^i}\frac{\partial x_r}{\partial \theta^j} \quad , \quad G_{ij} = \underset{\sim}{G}_i \cdot \underset{\sim}{G}_j = \frac{\partial y^r}{\partial \theta^i}\frac{\partial y_r}{\partial \theta^j} \ , \tag{9}$$

$$g^{ir} g_{rj} = \delta^i_j \ , \ G^{ir} G_{rj} = \delta^i_j \ ,$$

transformation between 2 sets of base vectors

$$\underset{\sim}{g}^i = g^{ir} \underset{\sim}{g}_r \ , \quad \underset{\sim}{G}^i = G^{ir} \underset{\sim}{G}_r \ ,$$

where δ^i_j denotes the Kronecker symbol. If

$$g = |g_{ij}| \ , \tag{10}$$

Jacobian

we see that

$$\sqrt{g} = \left| \frac{\partial x^i}{\partial \theta^j} \right| \ , \tag{11}$$

the positive root being chosen in view of (3).

The displacement vector $\underset{\sim}{v}(\theta^1, \theta^2, \theta^3; \tau)$ is defined by

$$\underset{\sim}{v} = \underset{\sim}{R} - \underset{\sim}{r} \tag{12}$$

and may be expressed as

$$\underset{\sim}{v} = v^i \underset{\sim}{g}_i = v_i \underset{\sim}{g}^i \ . \tag{13}$$

The strain tensor is defined to be

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right) , \tag{14}$$

and, in terms of the displacement components,

$$\gamma_{ij} = \frac{1}{2}\left(v_i|_j + v_j|_i + v^r|_i \, v_r|_j\right) \ , \tag{15}$$

where a single vertical line denotes covariant differentiation with respect to θ^i and g_{ij}, g^{ij}.

The stress vector $\underset{\sim}{{}_\circ t}$, per unit area of the undeformed body, associated with a surface in the deformed body, whose unit normal in its undeformed position is $\underset{\sim}{{}_\circ n}$, is

$$\underset{\sim}{{}_{\circ}t} = s^{ij}\,{}_{\circ}n_i\,\underset{\sim}{G}_j \;, \qquad (16)$$

$$\underset{\sim}{{}_{\circ}n} = {}_{\circ}n_i\,\underset{\sim}{g}^i \;.$$ comp. of t

The stress tensor s^{ij} is measured per unit area of the undeformed body while defining the state of stress in the deformed body. The equations of motion in terms of s^{ij} are

$$[s^{ir}(\delta^j_r + v^j|_r)]|_i + \rho_o\,{}_{\circ}F^j = \rho_o\,{}_{\circ}f^j \;, \qquad (17)$$

where

$$\underset{\sim}{F} = {}_{\circ}F^i\,\underset{\sim}{g}_i \quad , \quad \underset{\sim}{f} = {}_{\circ}f^i\,\underset{\sim}{g}_i \qquad (18)$$

are respectively the body force vector and the acceleration vector, and ρ_o is the density of the undeformed body. In the nonpolar case, s^{ij} is symmetric, i.e.,

$$\epsilon_{ijk}\,s^{ij} = 0 \;, \qquad (19)$$

where

$$\epsilon_{ijk} = \sqrt{g}\;e_{ijk} \;, \qquad (20)$$

e_{ijk} being the permutation symbol.

We have

$$s^{ij} = \frac{1}{2}\left(\frac{\partial\Sigma^*}{\partial\gamma_{ij}} + \frac{\partial\Sigma^*}{\partial\gamma_{ji}}\right) \;, \qquad (21)$$

where, for an elastic body, Σ^* is the strain energy function, measured per unit volume of the undeformed body and has the property

$$\delta\Sigma^* = s^{ij}\,\delta\gamma_{ij} \;, \qquad (22)$$

where δ is the symbol denoting the variation of a function.

If the stress vector $\underset{\sim}{{}_{\circ}t}$ be referred to base vectors $\underset{\sim}{g}_i$ in the undeformed body,

$$\underset{\sim}{{}_{\circ}t} = {}_{\circ}n_i\,t^{ij}\,\underset{\sim}{g}_j \;, \qquad (23)$$

where

$$t^{ij} = s^{ir}(\delta^j_r + v^j|_r) \tag{24}$$

is another stress tensor measured per unit area of the undeformed body, depicting the state of stress of the deformed body.

The equations of motion in terms of t^{ij} are

$$t^{ij}|_i + \rho_o\, {}_oF^j = \rho_o\, {}_of^j , \tag{25}$$

where t^{ij} is not symmetric but satisfies, in the nonpolar case,

$$t^{im} \underset{\sim}{g}_m \cdot \underset{\sim}{G}^j = t^{jm} \underset{\sim}{g}_m \cdot \underset{\sim}{G}^i . \tag{26}$$

We have

$$t^{ij} = \frac{\partial \Sigma^*}{\partial v_j|_i} . \tag{27}$$

The development of these particular aspects of the measure of stress can be traced with the help of the references in [2a] and [3]. In addition, in the course of our study, several references to previous work in terms of the stress tensors s^{ij} and t^{ij}, within three- and two-dimensional theories, are given. We note here that some of the related basic equations of three-dimensional elasticity have also been discussed, for instance, in [4], [5], [6], [7], [8a] for both the static and dynamic cases, the last reference offering a very general treatment. The notation and terminology vary (cf. Appendix). We follow mainly [1] where a concise account can be found.

As a more specific set of stress-strain relations than (21), we may assume

$$s^{ij} = C^{ijrs}(\gamma_{rs} - \alpha_{rs}\Theta) , \tag{28}$$

where, for an anisotropic material, in the presence of a prescribed steady temperature field $\Theta(\theta^1, \theta^2, \theta^3)$, the C^{ijrs} are isothermal

stiffnesses and the α_{ij} are strain-temperature coefficients at constant stress. The following symmetry relations

$$C^{ijrs} = C^{jirs} = C^{ijsr} = C^{rsij} , \qquad \alpha_{ij} = \alpha_{ji} \tag{29}$$

are satisfied.

For a medium having elastic symmetry with respect to the surface θ^3 = const., equations (28) reduce to

$$\begin{aligned} s^{\alpha\beta} &= C^{\alpha\beta\delta\nu}(\gamma_{\delta\nu} - \alpha_{\delta\nu}\Theta) + C^{\alpha\beta 33}(\gamma_{33} - \alpha_{33}\Theta) , \\ s^{\alpha 3} &= 2\, C^{\alpha 3\beta 3}(\gamma_{\beta 3} - \alpha_{\beta 3}\Theta) , \\ s^{33} &= C^{33\delta\nu}(\gamma_{\delta\nu} - \alpha_{\delta\nu}\Theta) + C^{3333}(\gamma_{33} - \alpha_{33}\Theta) . \end{aligned} \tag{30}$$

Such linear relations as (28), less the temperature terms, have been offered before, in the isotropic case, within a theory of "infinitesimal strain but large displacement gradients and rotations" which is valid for thin bodies, and does not imply a linearized version of the strain-displacement relations. The history and a critical review of this approximation to the nonlinear stress-strain relations that can be obtained from a power series expansion of the strain energy function--implying isothermal or adiabatic deformation--assumed to be analytic, for isotropic bodies, is given in [9]. Using the conventional representation of stress, relations similar to (30), but with $\Theta = 0$, have been employed for shells in [10a] on the basis of [11]. Expressions equivalent to (28) can be found in [12].

The general variational principle that later will be employed in deriving the plate and shell equations is given in the next subsection.

1.2. Variational Principle

The Hellinger-Reissner principle discussed in [2b] on the basis of [13] and [3] leads to Cauchy's first law of motion and the mixed boundary conditions of the theory of elasticity in terms of a reference state, within a two point field description. The same is also illustrated in [14a]. Reference [15], as given in [16], contains, according to [3], an independent derivation of the results in [3]. We have not been able to examine [13] and [15].

Again, for the sake of a self-contained presentation, it is our purpose in this section to formulate the principle in convected general curvilinear coordinates, using the stress tensor $\overset{\cdot\cdot}{s}{}^{ij}$, and in such a manner as to obtain from it the strain-displacement relations (15), by following a procedure introduced in [16] and [17].

The following notation will be employed.

Let ${}_{o}V$ be the volume of the undeformed body, and ${}_{o}A_s$ and ${}_{o}A_v$ be the two parts of its total boundary where the stress and displacement vectors are prescribed, respectively. Let dV and dA denote the corresponding elements of volume and area, respectively. Let $\dot{s}_{*}^{i}$ be the components of the stress vector ${}_{o}\underset{\sim}{t}$ referred to base vectors in the undeformed body. Let tildes indicate a prescribed quantity.

Then, rephrasing the version given in [2b], the <u>modified Hellinger-Reissner theorem</u> asserts that the <u>variational principle</u>

$$\begin{aligned}\int_{{}_oV} \underset{o}{\rho}({}_of^i - {}_oF^i)\delta v_i \, dV = \delta\Big[\int_{{}_oV} (s^{ij}\gamma_{ij} - \Sigma^*)\, dV - \\ - \int_{{}_oV} \tfrac{1}{2} s^{ij}(v_i|_j + v_j|_i + v^r|_i v_r|_j)\, dV + \int_{{}_oA_s} \tilde{s}_*^{\,i} v_i \, dA + \\ + \int_{{}_oA_v} s_*^i (v_i - \tilde{v}_i)\, dA\Big] \quad ,\end{aligned} \tag{31}$$

<u>where γ_{ij}, s^{ij}, v_i and s_*^i are varied independently, is equivalent to Cauchy's first law in ${}_oV$, to the stress boundary condition on the part ${}_oA_s$ of the boundary, to the displacement boundary condition on the remaining part ${}_oA_v$, and to the stress-strain and strain-displacement relations in ${}_oV$, when the symmetries of γ_{ij} and s^{ij} are both used</u>.

To establish this theorem, we carry out the indicated variation in (31). Using Green's transformation and combining the resulting volume and surface integrals, we obtain

$$\begin{aligned}
&\int_{{}_oV}\Big\langle\Big[s^{ij}-\tfrac{1}{2}\Big(\frac{\partial\Sigma^*}{\partial\gamma_{ij}}+\frac{\partial\Sigma^*}{\partial\gamma_{ji}}\Big)\Big]\delta\gamma_{ij}+\Big[\gamma_{ij}-\\
&-\tfrac{1}{2}(v_i|_j+v_j|_i+v^r|_i\, v_r|_j)\Big]\delta s^{ij}+\\
&+\{[s^{jr}(\delta^i{}_r+v^i|_r)]|_j+\rho_o({}_oF^i-{}_of^i)\}\delta v_i\Big\rangle dV+\\
&+\int_{{}_oA_s}[\tilde{s}_*^i-{}_on_j\, s^{jr}(\delta^i{}_r+v^i|_r)]\delta v_i\, dA+\\
&+\int_{{}_oA_v}\{[s_*^i-{}_on_j\, s^{jr}(\delta^i{}_r+v^i|_r)]\delta v_i+(v_i-\tilde{v}_i)\delta s_*^i\}dA=0.
\end{aligned}\tag{32}$$

For independent and arbitrary variations of the indicated quantities, equations (21), (15), and (17) follow, in ${}_oV$, while on ${}_oA_s$

$$\tilde{s}_*^i = s_*^i = {}_on_j\, s^{jr}(\delta^i{}_r+v^i|_r)\,,\tag{33}$$

and on ${}_oA_v$

$$\tilde{v}_i = v_i\ .\tag{34}$$

The theorem is thus verified.

The theorem given in [3] now follows from (31), by using the inverse

$$\begin{aligned}
\gamma_{ij} &= \tfrac{1}{2}\Big(\frac{\partial W}{\partial s_{ij}}+\frac{\partial W}{\partial s_{ji}}\Big),\\
\Sigma^* &= s^{ij}\gamma_{ij}-W
\end{aligned}\tag{35}$$

of the transformation

$$W = s^{ij}\gamma_{ij} - \Sigma^* \,,$$
$$s^{ij} = \frac{1}{2}\left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}}\right) \,, \tag{36}$$

provided the Hessian of Σ^* does not vanish and the strain-displacement relations (15) are imposed a priori. Thus, W is the complementary energy function per unit volume of the undeformed body.

For further contributions on this matter, we refer the reader to [18] where still another formulation, discussed in [14b], is given, as well as to [19]. We have not been able to examine [18].

Obviously, to various stages of linearization in the strain-displacement relations may correspond simplified versions of (31), and hence, of (17) and (33). Recently, a reformulation of (31) in terms of the elongation and mean rotation tensors, has been given in [20] where some of the gradual linearizations of the strain-displacement relations expressed in terms of the latter tensors, earlier available in the literature, have also been discussed.

Some preliminaries for plates and shells follow in the next two subsections.

1.3. Preliminaries for Plates

When referring to the plate, the original set of convected general curvilinear coordinates θ^i will be identified with a set of convected normal coordinates--the middle plane of the plate, $\theta^3 = 0$, being the reference plane--so that the corresponding metric tensor of the undeformed plate space is given by

$$g_{\alpha\beta} = a_{\alpha\beta}(\theta^1,\theta^2) \,, \quad g_{\alpha 3} = a_{\alpha 3} = 0 \,, \quad g_{33} = a_{33} = 1 \,, \tag{37}$$

where a_{ij} is the symmetric metric tensor of the middle plane of the

undeformed plate. The θ^{α}- curves are the coordinate curves forming a system of curvilinear coordinates on the middle plane of the plate. On account of (37), a single vertical line will now denote covariant differentiation with respect to θ^{i} and a_{ij}, a^{ij}, keeping in mind the new meaning of θ^{i} and the fact that, since the coordinate curves span a plane, the order of covariant differentiation is now immaterial.

The undeformed plate of uniform thickness $2h$ is defined as the region of space bounded by the two plane faces $\theta^3 = +h$ and $\theta^3 = -h$, symmetrically disposed with respect to the middle plane, ${}_0a$, and the edge boundary, a cylindrical surface which intersects the middle plane along a simple closed curve, ${}_0c$, and whose generators lie along the normal to the middle plane. A simply connected plate will be assumed. No singularities of any kind are supposed to be present.

In order to illustrate our two methods of deriving the fundamental equations of plates in terms of a reference state, the displacement components will be taken as

$$
\begin{aligned}
v_{\alpha} &= u_{\alpha}(\theta^1, \theta^2; \tau) + \theta^3 \psi_{\alpha}(\theta^1, \theta^2; \tau) , \\
v_3 &= w(\theta^1, \theta^2; \tau) + \theta^3 w_1(\theta^1, \theta^2; \tau) .
\end{aligned}
\tag{38}
$$

From (38) and (15)

$$
\begin{aligned}
\gamma_{\alpha\beta} &= {}_0\gamma_{\alpha\beta} + \theta^3\, {}_1\gamma_{\alpha\beta} + (\theta^3)^2\, {}_2\gamma_{\alpha\beta} , \\
\gamma_{\alpha 3} &= {}_0\gamma_{\alpha 3} + \theta^3\, {}_1\gamma_{\alpha\beta} , \\
\gamma_{33} &= {}_0\gamma_{33} ,
\end{aligned}
\tag{39}
$$

where ${}_0\gamma_{\alpha\beta}$, ${}_1\gamma_{\alpha\beta}$, ${}_2\gamma_{\alpha\beta}$, ${}_0\gamma_{\alpha 3}$, ${}_1\gamma_{\alpha 3}$, ${}_0\gamma_{33}$ are independent of θ^3 , and can be evaluated in terms of the displacement functions introduced in (38), by substituting (38) into (15).

We introduce the following definitions for the stress resultants per unit length of coordinate curves on ${}_0a$ and effective external loads per unit area of ${}_0a$.

$$\begin{aligned}
&N^{\alpha\beta} = N^{\beta\alpha} = \int_{-h}^{+h} s^{\alpha\beta}\, d\theta^3\,, \quad M^{\alpha\beta} = M^{\beta\alpha} = \int_{-h}^{+h} \theta^3 s^{\alpha\beta}\, d\theta^3,\\
&K^{\alpha\beta} = K^{\beta\alpha} = \int_{-h}^{+h} (\theta^3)^2 s^{\alpha\beta}\, d\theta^3, \quad Q^{\alpha} = \int_{-h}^{+h} s^{\alpha 3}\, d\theta^3,\\
&T^{\alpha} = \int_{-h}^{+h} \theta^3 s^{\alpha 3}\, d\theta^3\,, \qquad N^{33} = \int_{-h}^{+h} s^{33}\, d\theta^3\,,\\
&\mathcal{F}^{\alpha} = \int_{-h}^{+h} \rho_0\, {}_0F^{\alpha}\, d\theta^3\,, \qquad \mathcal{M}^{\alpha} = \int_{-h}^{+h} \theta^3 \rho_0\, {}_0F^{\alpha}\, d\theta^3,\\
&\mathcal{F} = \int_{-h}^{+h} \rho_0\, {}_0F^{3}\, d\theta^3, \qquad \mathcal{M} = \int_{-h}^{+h} \theta^3 \rho_0\, {}_0F^{3}\, d\theta^3,\\
&\mathcal{f}^{\alpha} = \int_{-h}^{+h} \rho_0\, {}_0f^{\alpha}\, d\theta^3, \qquad c^{\alpha} = \int_{-h}^{+h} \theta^3 \rho_0\, {}_0f^{\alpha}\, d\theta^3,\\
&\mathcal{f} = \int_{-h}^{+h} \rho_0\, {}_0f^{3}\, d\theta^3, \qquad c = \int_{-h}^{+h} \theta^3 \rho_0\, {}_0f^{3}\, d\theta^3,\\
&p^{\alpha} = \mathcal{F}^{\alpha} + \left[s^{\beta 3}(k^{\alpha}_{\beta} + \theta^3 \psi^{\alpha}|_{\beta}) + s^{33}\psi^{\alpha} \right]_{-h}^{+h}\,,\\
&p = \mathcal{F} + \left[s^{\alpha 3}(w_{,\alpha} + \theta^3 w_{1,\alpha}) + s^{33}(1 + w_1) \right]_{-h}^{+h}\,,\\
&m^{\alpha} = \mathcal{M}^{\alpha} + \left[\theta^3 s^{\beta 3}(k^{\alpha}_{\beta} + \theta^3 \psi^{\alpha}|_{\beta}) + \theta^3 s^{33}\psi^{\alpha} \right]_{-h}^{+h}\,,\\
&m = \mathcal{M} + \left[\theta^3 s^{\alpha 3}(w_{,\alpha} + \theta^3 w_{1,\alpha}) + \theta^3 s^{33}(1 + w_1) \right]_{-h}^{+h},\\
&k^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + u^{\alpha}|_{\beta}\,.
\end{aligned} \tag{40}$$

Generally, it may be possible to formulate the stress-strain relations for the resultant stresses in the plate, as a two-dimensional analog of (21), in terms of an arbitrary strain energy function Σ, per unit area of ${}_0a$, and defined by

$$\Sigma = \int_{-h}^{+h} \Sigma^*\, d\theta^3\,. \tag{41}$$

The method is then similar to that developed for shells in [21] where the conventional measure of stress and Kirchhoff's approximation have been employed, while the strain energy function is formally but tacitly symmetrized.

Thus, from (22)

$$\int_{{}_0V} s^{ij}\,\delta\gamma_{ij}\,dV = \int_{{}_0V} \delta\Sigma^*\,dV\,, \tag{42}$$

where

$$\begin{aligned} dV &= \sqrt{a}\;d\theta^1 d\theta^2 d\theta^3 \\ &= d\theta^3\,dA\,, \\ a &= |a_{\alpha\beta}|\,, \end{aligned} \tag{43}$$

we obtain, by integration across the thickness of the plate

$$\begin{aligned} &\int_{{}_0a} \left(N^{\alpha\beta}\delta_0\gamma_{\alpha\beta} + M^{\alpha\beta}\delta_1\gamma_{\alpha\beta} + K^{\alpha\beta}\delta_2\gamma_{\alpha\beta} + 2Q^{\alpha}\delta_0\gamma_{\alpha 3} + \right. \\ &\left. + 2T^{\alpha}\delta_1\gamma_{\alpha 3} + N^{33}\delta_0\gamma_{33} \right) dA = \\ &= \int_{{}_0a} \left[\frac{1}{2}\left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_0\gamma_{\beta\alpha}} \right)\delta_0\gamma_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_1\gamma_{\beta\alpha}} \right)\delta_1\gamma_{\alpha\beta} + \right. \\ &+ \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_2\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_2\gamma_{\beta\alpha}} \right)\delta_2\gamma_{\alpha\beta} + \left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha 3}} + \frac{\partial\Sigma}{\partial_0\gamma_{3\alpha}} \right)\delta_0\gamma_{\alpha 3} + \\ &\left. + \left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha 3}} + \frac{\partial\Sigma}{\partial_1\gamma_{3\alpha}} \right)\delta_1\gamma_{\alpha 3} + \frac{\partial\Sigma}{\partial_0\gamma_{33}}\delta_0\gamma_{33} \right] dA\,, \end{aligned} \tag{44}$$

where (39), (40), and the symmetries of the stress and strain tensors have been employed. We conclude

$$\begin{aligned} N^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_0\gamma_{\beta\alpha}} \right), & M^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_1\gamma_{\beta\alpha}} \right), \\ K^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_2\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial_2\gamma_{\beta\alpha}} \right), & Q^{\alpha} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha 3}} + \frac{\partial\Sigma}{\partial_0\gamma_{3\alpha}} \right), \\ T^{\alpha} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha 3}} + \frac{\partial\Sigma}{\partial_1\gamma_{3\alpha}} \right), & N^{33} &= \frac{\partial\Sigma}{\partial_0\gamma_{33}}\,, \end{aligned} \tag{45}$$

which expressions, for arbitrary Σ and a strain distribution such as (39), constitute nonlinear resultant stress-strain relations for the plate.

1.4. Preliminaries for Shells

When referring to the shell, the original set of convected general curvilinear coordinates θ^i will be identified with a set of convected normal coordinates--the middle surface of the shell, $\theta^3 = 0$, being taken as the reference surface--so that the corresponding metric tensor of the undeformed shell space is given by

$$
\begin{aligned}
g_{\alpha\beta} &= \mu^\delta_\alpha \, \mu^\nu_\beta \, a_{\delta\nu} \, , \\
g^{\alpha\beta} &= (\mu^{-1})^\alpha_\delta \, (\mu^{-1})^\beta_\nu \, a^{\delta\nu} , \\
g_{\alpha 3} &= 0 \quad , \qquad g_{33} = 1 \quad ,
\end{aligned}
\tag{46}
$$

where

$$\mu^\alpha_\beta = \delta^\alpha_\beta - \theta^3 b^\alpha_\beta \tag{47}$$

is, in the notation of [10b], the inverse of the "shell-tensor," $(\mu^{-1})^\alpha_\beta$, introduced in [22], and such that

$$\mu^\beta_\delta \, (\mu^{-1})^\alpha_\beta = \delta^\alpha_\delta \; . \tag{48}$$

The coefficients of the first and second fundamental forms of the undeformed middle surface of the shell are denoted by $a_{\alpha\beta}(\theta^1,\theta^2)$ and $b_{\alpha\beta}(\theta^1,\theta^2)$, respectively. The corresponding third fundamental form is given by

$$
\begin{aligned}
c_{\alpha\beta} &= b_{\alpha\delta} \, b^\delta_\beta \\
&= 2H \, b_{\alpha\beta} - \frac{b}{a} \, a_{\alpha\beta} \quad , \\
H &= \tfrac{1}{2} \, b^\alpha_\alpha \quad , \quad b = |b_{\alpha\beta}| \; .
\end{aligned}
\tag{49}
$$

In the case of plates, μ^α_β is simply the Kronecker symbol, since then $b_{\alpha\beta} = 0$.

A single vertical line denotes covariant differentiation with respect to the convected normal coordinates θ^i and g_{ij}, g^{ij}.

As remarked in [10c], μ^{α}_{β} and its inverse, the "shell-tensor," act as "shifters" in our space of normal coordinates. Accordingly, they are used in obtaining the "shifted" tensor corresponding to a given tensor and vice versa.

We record for future use

$$\epsilon_{\alpha\beta 3} = \mu^{\delta}_{\alpha}\,\mu^{\nu}_{\beta}\,\bar{\epsilon}_{\delta\nu} \quad . \tag{50}$$

The undeformed shell of thickness $2h$ is the region of space bounded by the two faces, $\theta^3 = +h$ and $\theta^3 = -h$, symmetrically disposed with respect to the middle surface, ${}_0a$, and the edge boundary, a surface of revolution which intersects ${}_0a$ along a simple closed curve ${}_0\mathcal{C}$, and whose generators lie along the normal to ${}_0a$. A simply connected shell is assumed, and no singularities of any kind are supposed to be present.

In order to illustrate our two methods of deriving the fundamental equations of shells in terms of a reference state, the "shifted" displacement components, denoted by a bar, will be taken as

$$\begin{aligned} \bar{V}_{\alpha} &= u_{\alpha}(\theta^1,\theta^2;\tau) + \theta^3\psi_{\alpha}(\theta^1,\theta^2;\tau)\,, \\ \bar{V}_3 &= w(\theta^1,\theta^2;\tau) + \theta^3 w_1(\theta^1,\theta^2;\tau)\,. \end{aligned} \tag{51}$$

Substitution of (51) into (15) yields the following distribution of strains

$$\begin{aligned} \gamma_{\alpha\beta} &= {}_0\gamma_{\alpha\beta} + \theta^3\,{}_1\gamma_{\alpha\beta} + (\theta^3)^2\,{}_2\gamma_{\alpha\beta}\,, \\ \gamma_{\alpha 3} &= {}_0\gamma_{\alpha 3} + \theta^3\,{}_1\gamma_{\alpha 3}\,, \\ \gamma_{33} &= {}_0\gamma_{33} \end{aligned} \tag{52}$$

which is identical to (39) although ${}_0\gamma_{\alpha\beta}$, ${}_1\gamma_{\alpha\beta}$, ${}_2\gamma_{\alpha\beta}$, ${}_0\gamma_{\alpha 3}$, ${}_1\gamma_{\alpha 3}$, ${}_0\gamma_{33}$ are now more complicated functions of the displacement functions introduced in (51).

The following relations are adopted from general formulae given in [10c].

$$
\begin{aligned}
&v_\alpha = \mu^\beta_\alpha \bar{v}_\beta \quad , \qquad v_3 = \bar{v}_3 \quad , \\
&v_\alpha|_\beta = \mu^\delta_\alpha (\bar{v}_\delta\|_\beta - b_{\delta\beta}\bar{v}_3) \; , \\
&v_\alpha|_3 = \mu^\beta_\alpha \bar{v}_{\beta,3} \quad , \quad v_3|_\alpha = \bar{v}_{3,\alpha} + b^\beta_\alpha \bar{v}_\beta \quad , \\
&v_3|_3 = \bar{v}_{3,3} \quad .
\end{aligned}
\tag{53}
$$

A double vertical line denotes covariant differentiation with respect to θ^α and a_{ij}, a^{ij} the order of covariant differentiation being important. A comma stands for partial differentiation with respect to θ^i.

Using (24), (46)-(48), (51), and (53) we obtain

$$
\begin{aligned}
t^{\alpha\beta} &= s^{\alpha\delta}\{\delta^\beta_\delta + (\mu^{-1})^\beta_\nu [u^\nu\|_\delta - b^\nu_\delta w + \theta^3(\psi^\nu\|_\delta - b^\nu_\delta w_1)]\} + \\
&\quad + s^{\alpha 3}(\mu^{-1})^\beta_\delta \psi^\delta \quad , \\
t^{3\alpha} &= s^{3\beta}\{\delta^\alpha_\beta + (\mu^{-1})^\alpha_\delta [u^\delta\|_\beta - b^\delta_\beta w + \theta^3(\psi^\delta\|_\beta - b^\delta_\beta w_1)]\} + \\
&\quad + s^{33}(\mu^{-1})^\alpha_\beta \psi^\beta \quad , \\
t^{\alpha 3} &= s^{\alpha\beta}[w_{,\beta} + b^\delta_\beta u_\delta + \theta^3(w_{1,\beta} + b^\delta_\beta \psi_\delta)] + s^{\alpha 3}(1+w_1) \; , \\
t^{33} &= s^{3\alpha}[w_{,\alpha} + b^\beta_\alpha u_\beta + \theta^3(w_{1,\alpha} + b^\beta_\alpha \psi_\beta)] + s^{33}(1+w_1) \; .
\end{aligned}
\tag{54}
$$

With the same meaning mentioned above attached to a bar, these relations are also taken from the corresponding general formulae in [10c], slightly modified for our purpose.

$$
\begin{aligned}
&\bar{t}^{\alpha\beta} = \mu^{\alpha}_{\nu}\,\mu^{\beta}_{\delta}\,t^{\nu\delta} \quad , \quad \bar{t}^{\alpha 3} = \mu^{\alpha}_{\beta}\,t^{\beta 3} \quad , \\
&\bar{t}^{3\alpha} = \mu^{\alpha}_{\beta}\,t^{3\beta} \quad , \quad \bar{t}^{33} = t^{33} \quad , \\
&t^{\alpha\beta}|_{\delta} = (\mu^{-1})^{\alpha}_{\nu}(\mu^{-1})^{\beta}_{\lambda}\left(\bar{t}^{\nu\lambda}\|_{\delta} - b^{\lambda}_{\delta}\,\bar{t}^{\nu 3} - b^{\nu}_{\delta}\,\bar{t}^{3\lambda}\right) , \\
&t^{\alpha 3}|_{\beta} = (\mu^{-1})^{\alpha}_{\delta}\left(\bar{t}^{\delta 3}\|_{\beta} + b^{\nu}_{\beta}\,\bar{t}^{\delta}_{\ \nu} - b^{\delta}_{\beta}\,\bar{t}^{33}\right) , \\
&t^{3\alpha}|_{3} = (\mu^{-1})^{\alpha}_{\beta}\,\bar{t}^{3\beta}{}_{,3} \quad , \quad \mu = |\mu^{\alpha}_{\beta}| \ , \\
&\mu\|_{\alpha} = \mu(\mu^{-1})^{\beta}_{\delta}\,\mu^{\delta}_{\alpha}\|_{\beta} \ , \quad \mu_{,3} = -\mu(\mu^{-1})^{\alpha}_{\beta}\,b^{\beta}_{\alpha} \ .
\end{aligned}
\tag{55}
$$

Equation $(55)_9$ involves the use of the Mainardi-Codazzi relations from differential geometry. In the case of plates, $\mu = 1$. The last equation in (55) is not explicitly given in [10c].

By a suitable combination of the relations summarized in (55), we obtain

$$
\begin{aligned}
&\mu\,\mu^{\delta}_{\beta}\,t^{\alpha\beta}|_{\alpha} = (\mu\,\mu^{\delta}_{\beta}\,t^{\alpha\beta})\|_{\alpha} - \mu\,\mu^{\delta}_{\alpha}(\mu^{-1})^{\beta}_{\nu}\,b^{\nu}_{\beta}\,t^{3\alpha} - \\
&\qquad - \mu\,b^{\delta}_{\alpha}\,t^{\alpha 3} \quad , \\
&\mu\,t^{\alpha 3}|_{\alpha} = (\mu\,t^{\alpha 3})\|_{\alpha} + \mu\,\mu^{\beta}_{\delta}\,b_{\beta\alpha}\,t^{\alpha\delta} - \mu(\mu^{-1})^{\alpha}_{\beta}\,b^{\beta}_{\alpha}\,t^{33} , \\
&\mu^{\beta}_{\alpha}\,t^{3\alpha}|_{3} = (\mu^{\beta}_{\alpha}\,t^{3\alpha})_{,3} \quad .
\end{aligned}
\tag{56}
$$

We introduce the following definitions for the stress resultants per unit length of coordinate curves on ${}_{o}\mathcal{a}$, and effective external loads per unit area of ${}_{o}\mathcal{a}$.

$$
\begin{aligned}
&N^{\alpha\beta} = \int_{-h}^{+h} \mu\,\mu^{\beta}_{\delta}\,s^{\alpha\delta}\,d\theta^3 \ , \quad M^{\alpha\beta} = \int_{-h}^{+h} \mu\,\mu^{\beta}_{\delta}\,s^{\alpha\delta}\,\theta^3\,d\theta^3 , \\
&Q^{\alpha} = \int_{-h}^{+h} \mu\,s^{\alpha 3}\,d\theta^3 \ , \quad T^{\alpha} = \int_{-h}^{+h} \mu\,s^{\alpha 3}\,\theta^3\,d\theta^3 \ , \\
&N^{33} = \int_{-h}^{+h} \mu\,s^{33}\,d\theta^3 \ , \quad 'N^{\alpha\beta} = {'N}^{\beta\alpha} = \int_{-h}^{+h} \mu\,s^{\alpha\beta}\,d\theta^3 , \\
&'M^{\alpha\beta} = {'M}^{\beta\alpha} = \int_{-h}^{+h} \mu\,s^{\alpha\beta}\,\theta^3\,d\theta^3 \ , \quad 'K^{\alpha\beta} = {'K}^{\beta\alpha} = \int_{-h}^{+h} \mu\,s^{\alpha\beta}(\theta^3)^2\,d\theta^3 ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}^{\alpha} &= \int_{-h}^{+h} \mu \mu^{\alpha}_{\beta} \rho_{o} {}_{o}F^{\beta} d\theta^{3}; \quad \mathcal{F} = \int_{-h}^{+h} \mu \rho_{o} {}_{o}F^{3} d\theta^{3}, \\
\mathcal{M}^{\alpha} &= \int_{-h}^{+h} \mu \mu^{\alpha}_{\beta} \rho_{o} {}_{o}F^{\beta} \theta^{3} d\theta^{3}, \quad \mathcal{M} = \int_{-h}^{+h} \mu \rho_{o} {}_{o}F^{3} \theta^{3} d\theta^{3}, \\
\not{f}^{\alpha} &= \int_{-h}^{+h} \mu \mu^{\alpha}_{\beta} \rho_{o} {}_{o}f^{\beta} d\theta^{3}, \quad \not{f} = \int_{-h}^{+h} \mu \rho_{o} {}_{o}f^{3} d\theta^{3}, \\
c^{\alpha} &= \int_{-h}^{+h} \mu \mu^{\alpha}_{\beta} \rho_{o} {}_{o}f^{\beta} \theta^{3} d\theta^{3}, \quad m = \int_{-h}^{+h} \mu \rho_{o} {}_{o}f^{3} \theta^{3} d\theta^{3},
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
p^{\alpha} &= \mathcal{F}^{\alpha} + \{ \mu \mu^{\alpha}_{\beta} s^{3\beta} + \mu s^{3\beta} [u^{\alpha}\|_{\beta} - b^{\alpha}_{\beta} w + \\
&\quad + \theta^{3} (\psi^{\alpha}\|_{\beta} - b^{\alpha}_{\beta} w_{1})] + \mu s^{33} \psi^{\alpha} \}_{-h}^{+h} \quad , \\
p &= \mathcal{F} + \{ \mu s^{3\alpha} [w_{,\alpha} + b^{\beta}_{\alpha} u_{\beta} + \theta^{3} (w_{1,\alpha} + b^{\beta}_{\alpha} \psi_{\beta})] + \\
&\quad + \mu s^{33} (1 + w_{1}) \}_{-h}^{+h} \quad , \\
m^{\alpha} &= \mathcal{M}^{\alpha} + \{ \theta^{3} \mu \mu^{\alpha}_{\beta} s^{3\beta} + \mu s^{3\beta} \theta^{3} [u^{\alpha}\|_{\beta} - b^{\alpha}_{\beta} w + \\
&\quad + \theta^{3} (\psi^{\alpha}\|_{\beta} - b^{\alpha}_{\beta} w_{1})] + \theta^{3} \mu s^{33} \psi^{\alpha} \}_{-h}^{+h} \quad , \\
m &= \mathcal{M}^{3} + \{ \theta^{3} \mu s^{3\alpha} [w_{,\alpha} + b^{\beta}_{\alpha} u_{\beta} + \theta^{3} (w_{1,\alpha} + b^{\beta}_{\alpha} \psi_{\beta})] + \\
&\quad + \theta^{3} \mu s^{33} (1 + w_{1}) \}_{-h}^{+h} \quad , \\
k^{\alpha}_{\beta} &= \delta^{\alpha}_{\beta} + u^{\alpha}\|_{\beta} \quad .
\end{aligned}
$$

For shells, the presence of the symmetric stress, resultants indicated by a prime is of interest since the latter need not be used in the conventional theory barring special definitions. Clearly,

$$
\begin{aligned}
N^{\alpha\beta} &= {}'N^{\alpha\beta} - b^{\beta}_{\delta} \, {}'M^{\alpha\delta} \quad , \\
M^{\alpha\beta} &= {}'M^{\alpha\beta} - b^{\beta}_{\delta} \, {}'K^{\alpha\delta} \quad .
\end{aligned} \tag{58}
$$

The general stress-strain relations for shells can again be formulated as a two-dimensional analog of (21), in terms of an arbitrary strain energy function Σ, per unit area of ${}_{o}a$, and defined by

$$
\Sigma = \int_{-h}^{+h} \mu \, \Sigma^{*} d\theta^{3} \quad . \tag{59}
$$

A similar procedure to that adopted for plates in the previous section results in

$$'N^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial_0 \gamma_{\alpha\beta}} + \frac{\partial \Sigma}{\partial_0 \gamma_{\beta\alpha}}\right), \quad 'M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial_1 \gamma_{\alpha\beta}} + \frac{\partial \Sigma}{\partial_1 \gamma_{\beta\alpha}}\right),$$

$$'K^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial_2 \gamma_{\alpha\beta}} + \frac{\partial \Sigma}{\partial_2 \gamma_{\beta\alpha}}\right), \quad Q^{\alpha} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial_0 \gamma_{\alpha 3}} + \frac{\partial \Sigma}{\partial_0 \gamma_{3\alpha}}\right), \tag{60}$$

$$T^{\alpha} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial_1 \gamma_{\alpha 3}} + \frac{\partial \Sigma}{\partial_1 \gamma_{3\alpha}}\right), \quad N^{33} = \frac{\partial \Sigma}{\partial_0 \gamma_{33}},$$

which expressions, for arbitrary Σ and a strain distribution such as (52), constitute nonlinear stress-strain relations for the shell. From (58) and (60) follow nonlinear stress-strain relations for the unprimed resultants.

In the following sections, we develop the equations of plates and shells based on the information and intermediate results presented so far.

We first pass to the derivation of the plate equations of motion by integrating the three-dimensional stress equations of motion across the thickness of the undeformed plate.

II. EQUATIONS OF MOTION OF PLATES BY INTEGRATION

It is our purpose in this section to derive the equations of motion of plates in terms of a reference state by integrating the corresponding version of Cauchy's laws of motion across the thickness of the undeformed plate when the stress tensor s^{ij} is employed. An analogous procedure, based on the related equations involving t^{ij}, has been used in [23] in obtaining the equilibrium equations of plates.

Prior to integration with respect to θ^3, equations (17) are put into the equivalent form

$$
\begin{aligned}
&[s^{\beta\nu}(\delta^\alpha_\nu + a^{\alpha\delta} v_\delta|_\nu)]|_\beta + (s^{\beta 3} a^{\alpha\delta} v_{\delta,3})|_\beta + \\
&+ [s^{\beta 3}(\delta^\alpha_\beta + a^{\alpha\delta} v_\delta|_\beta)]_{,3} + (s^{33} a^{\alpha\beta} v_{\beta,3})_{,3} + \\
&+ \rho_0\,{}_0F^\alpha = \rho_0\,{}_0f^\alpha \ , \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
&(s^{\alpha\beta} v_{3,\beta})|_\alpha + [s^{\alpha 3}(1+v_{3,3})]|_\alpha + (s^{\alpha 3} v_{3,\alpha})_{,3} + \\
&+ [s^{33}(1+v_{3,3})]_{,3} + \rho_0\,{}_0F^3 = \rho_0\,{}_0f^3 \ , \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
&[\theta^3 s^{\beta\nu}(\delta^\alpha_\nu + a^{\alpha\delta} v_\delta|_\nu)]|_\beta + (\theta^3 s^{\beta 3} a^{\alpha\delta} v_{\delta,3})|_\beta + \\
&+ [\theta^3 s^{\beta 3}(\delta^\alpha_\beta + a^{\alpha\delta} v_\delta|_\beta)]_{,3} - s^{\beta 3}(\delta^\alpha_\beta + a^{\alpha\delta} v_\delta|_\beta) + \\
&+ (\theta^3 s^{33} a^{\alpha\beta} v_{\beta,3})_{,3} - s^{33} a^{\alpha\beta} v_{\beta,3} + \theta^3 \rho_0\,{}_0F^\alpha = \theta^3 \rho_0\,{}_0f^\alpha \ , \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
&(\theta^3 s^{\beta\alpha} v_{3,\alpha})|_\beta + [\theta^3 s^{\beta 3}(1+v_{3,3})]|_\beta + (\theta^3 s^{\alpha 3} v_{3,\alpha})_{,3} - \\
&- s^{\alpha 3} v_{3,\alpha} + [\theta^3 s^{33}(1+v_{3,3})]_{,3} - s^{33}(1+v_{3,3}) + \\
&+ \theta^3 \rho_0\,{}_0F^3 = \theta^3 \rho_0\,{}_0f^3 \ , \qquad (64)
\end{aligned}
$$

where (63) and (64) follow respectively from (61) and (62) upon multiplication by θ^3.

The system (61)-(64) can now be integrated across the thickness of the undeformed plate after substitution from equations (38) for the displacement components in terms of the displacement functions.

Using the definitions (40), the results of an integration with respect to the thickness coordinate are the following plate equations of motion in terms of a reference state.

$$(N^{\beta\delta} k^{\alpha}_{\delta})|_{\beta} + (M^{\beta\delta}\psi^{\alpha}|_{\delta})|_{\beta} + (Q^{\beta}\psi^{\alpha})|_{\beta} + p^{\alpha} = f^{\alpha} , \tag{65}$$

$$[Q^{\alpha}(1+w_1)]|_{\alpha} + (N^{\alpha\beta}w_{,\beta})|_{\alpha} + (M^{\alpha\beta}w_{1,\beta})|_{\alpha} + p = f , \tag{66}$$

$$\begin{aligned}(M^{\beta\delta}k^{\alpha}_{\delta})|_{\beta} - Q^{\beta}k^{\alpha}_{\beta} + (K^{\beta\delta}\psi^{\alpha}|_{\delta})|_{\beta} &+ \\ + (T^{\alpha}|_{\alpha} - N^{33})\psi^{\alpha} + m^{\alpha} &= c^{\alpha} ,\end{aligned} \tag{67}$$

$$\begin{aligned}(M^{\alpha\beta}w_{,\beta})|_{\alpha} + (K^{\alpha\beta}w_{1,\beta})|_{\alpha} - Q^{\alpha}w_{,\alpha} &+ \\ +(T^{\alpha}|_{\alpha} - N^{33})(1+w_1) + m &= c .\end{aligned} \tag{68}$$

It is of interest to note that a displacement distribution equivalent to

$$\begin{aligned} v_{\alpha} &= u_{\alpha}(\theta^1,\theta^2;\tau) + \theta^3\psi_{\alpha}(\theta^1,\theta^2;\tau) , \\ v_3 &= w(\theta^1,\theta^2;\tau) \end{aligned} \tag{69}$$

was used in [24] in deriving, from Hamilton's principle and in cartesian coordinates, plate equations of motion which do not involve the resultants $K^{\alpha\beta}$, T^{α}, N^{33}, their effect on the strain energy of the plate having been explicitly neglected at the outset. The equations of motion corresponding to (69) follow from (65)-(68) by putting $w_1 = 0$ in (66) and in the definition of p in (40), and dropping (68) and m altogether.

In the next section, we extend the method used here to shells.

III. EQUATIONS OF MOTION OF SHELLS BY INTEGRATION

In this section, we make use of the same approach employed for plates in the preceding one in order to obtain the equations of motion of shells in terms of a reference state, i.e., integration of the corresponding Cauchy's laws of motion across the thickness of the undeformed thin body.

Thus, we shall perform the integration of (17) and (19), and, in the process, make use of the relative simplicity of (25) and the relation (24).

From (25), (56), and (55), we obtain

$$(\mu\mu^{\delta}_{\beta}t^{\alpha\beta})\|_{\alpha} - \mu b^{\delta}_{\alpha}t^{\alpha 3} + (\mu\mu^{\delta}_{\alpha}t^{3\alpha})_{,3} + \\ + \mu\mu^{\delta}_{\alpha}\rho_{o\,o}F^{\alpha} = \mu\mu^{\delta}_{\alpha}\rho_{o\,o}f^{\alpha} \,, \tag{70}$$

$$(\mu t^{\alpha 3})\|_{\alpha} + \mu\mu^{\alpha}_{\beta}b_{\alpha\delta}t^{\delta\beta} + (\mu t^{33})_{,3} + \\ + \mu\rho_{o\,o}F^{3} = \mu\rho_{o\,o}f^{3} \,, \tag{71}$$

$$(\theta^3\mu\mu^{\delta}_{\beta}t^{\alpha\beta})\|_{\alpha} - \theta^3\mu b^{\delta}_{\alpha}t^{\alpha 3} + (\theta^3\mu\mu^{\delta}_{\alpha}t^{3\alpha})_{,3} - \\ - \mu\mu^{\delta}_{\alpha}t^{3\alpha} + \theta^3\mu\mu^{\delta}_{\alpha}\rho_{o\,o}F^{\alpha} = \theta^3\mu\mu^{\delta}_{\alpha}\rho_{o\,o}f^{\alpha} \,, \tag{72}$$

$$(\theta^3\mu t^{\alpha 3})\|_{\alpha} + \theta^3\mu\mu^{\alpha}_{\beta}b_{\alpha\delta}t^{\delta\beta} - \mu t^{33} + \\ + (\theta^3\mu t^{33})_{,3} + \theta^3\mu\rho_{o\,o}F^{3} = \theta^3\mu\rho_{o\,o}f^{3} \,, \tag{73}$$

where (72) and (73) follow respectively from (70) and (71) upon multiplication by θ^3.

The system (70)-(73) can now be integrated across the thickness of the undeformed shell, following the introduction of suitable stress and couple resultants as well as external effective loads, as in [10d] where the conventional stress equations of motion were used. We prefer,

however, to substitute (54) into (70)-(73) prior to the integration, in order to use the stress and couple resultants as well as external effective loads in terms of s^{ij} as defined in (57).

The integration of (70)-(73) following this substitution for the components of t^{ij} in terms of the components of s^{ij} and the displacement functions, in conjunction with the definitions (57), leads to equations containing both primed and unprimed, i.e., symmetric as well as asymmetric, stress and couple resultants. When (58) is employed, the results can be expressed in terms of the symmetric quantities only, and are here so presented. However, the essentially mixed character of the shell equations of motion should be kept in mind when the physical significance of the stress resultants is to be considered.

The equations of motion for a shell in terms of a reference state are

$$\begin{aligned} &[{}'N^{\alpha\beta}(k^{\delta}_{\beta} - b^{\delta}_{\beta}w)]\|_{\alpha} + \{{}'M^{\alpha\beta}[\psi^{\delta}\|_{\beta} - b^{\delta}_{\beta}(1+w_{1})]\}\|_{\alpha} + \\ &+ (Q^{\alpha}\psi^{\delta})\|_{\alpha} - {}'N^{\alpha\beta}(b^{\delta}_{\alpha}w_{,\beta} + b^{\delta}_{\alpha}b^{\nu}_{\beta}u_{\nu}) - \\ &- {}'M^{\alpha\beta}(b^{\delta}_{\alpha}w_{1,\beta} + b^{\delta}_{\alpha}b^{\nu}_{\beta}\psi_{\nu}) - Q^{\alpha}b^{\delta}_{\alpha}(1+w_{1}) + p^{\delta} = f^{\delta} \,, \end{aligned} \tag{74}$$

$$\begin{aligned} &[{}'M^{\alpha\beta}(k^{\delta}_{\beta} - b^{\delta}_{\beta}w)]\|_{\alpha} + \{{}'K^{\alpha\beta}[\psi^{\delta}\|_{\beta} - b^{\delta}_{\beta}(1+w_{1})]\}\|_{\alpha} + \\ &+ T^{\alpha}\|_{\alpha}\psi^{\delta} - Q^{\alpha}(k^{\delta}_{\alpha} - b^{\delta}_{\alpha}w) - N^{33}\psi^{\delta} - \\ &- {}'M^{\alpha\beta}(b^{\delta}_{\alpha}w_{,\beta} + b^{\delta}_{\alpha}b^{\nu}_{\beta}u_{\nu}) - {}'K^{\alpha\beta}(b^{\delta}_{\alpha}w_{1,\beta} + b^{\delta}_{\alpha}b^{\nu}_{\beta}\psi_{\nu}) + \\ &+ m^{\delta} = c^{\delta} \,, \end{aligned} \tag{75}$$

$$\begin{aligned} &[{}'N^{\alpha\beta}(w_{,\beta} + b^{\delta}_{\beta}u_{\delta})]\|_{\alpha} + [{}'M^{\alpha\beta}(w_{1,\beta} + b^{\delta}_{\beta}\psi_{\delta})]\|_{\alpha} + \\ &+ [Q^{\alpha}(1+w_{1})]\|_{\alpha} + {}'N^{\beta\delta}(b_{\alpha\beta}k^{\alpha}_{\delta} - c_{\beta\delta}w) + \\ &+ {}'M^{\beta\delta}[b_{\alpha\beta}\psi^{\alpha}\|_{\delta} - c_{\beta\delta}(1+w_{1})] + Q^{\beta}b_{\alpha\beta}\psi^{\alpha} + \\ &+ p = f \,, \end{aligned} \tag{76}$$

$$
\begin{aligned}
&[{}'M^{\alpha\beta}(w_{,\beta}+b^{\delta}_{\beta}u_{\delta})]\|_{\alpha} + [{}'K^{\alpha\beta}(w_{,\beta}+b^{\delta}_{\beta}\psi_{\delta})]\|_{\alpha} - \\
&- Q^{\alpha}(w_{,\alpha}+b^{\beta}_{\alpha}u_{\beta}) + (T^{\alpha}\|_{\alpha} - N^{33})(1+w_{1}) + \\
&+ {}'M^{\beta\delta}(b_{\alpha\beta}k^{\alpha}_{\delta} - c_{\beta\delta}w) + {}'K^{\beta\delta}[b_{\alpha\beta}\psi^{\alpha}\|_{\delta} - c_{\beta\delta}(1+w_{1})] + \\
&+ m = c \quad ,
\end{aligned} \tag{77}
$$

where we have used (49).

Finally, multiplying (19) by μ, and integrating across the thickness of the undeformed shell for $k=3$, we find

$$
\bar{\epsilon}_{\alpha\beta}\,(N^{\beta\alpha} - b^{\beta}_{\delta}M^{\delta\alpha}) = 0 \quad , \tag{78}
$$

where we have used (50). The other values of k lead to an identity. Substituting (58) into (78), we obtain

$$
\bar{\epsilon}_{\alpha\beta}\,({}'N^{\beta\alpha} + b^{\beta}_{\delta}b^{\alpha}_{\nu}\,{}'K^{\delta\nu}) = 0 \tag{79}
$$

in terms of the symmetric resultants.

The system of equations (74)-(77) and (79), with (58) in mind, constitute the equations of motion of shells in the reference state as obtained by integration of the corresponding three-dimensional equations across the thickness of the undeformed shell. Hamilton's principle is again used in [25], where cylindrical shells only are considered, with results which, on account of certain simplifying assumptions, are less comprehensive.

We note that equations (74)-(77) agree with the system (65)-(68) when all terms involving the coefficients of the second and hence, third fundamental form of the undeformed middle surface of the shell are made to vanish as in the case of plates.

In the following sections, the fundamental equations of plates and shells in terms of a reference state will be obtained from the variational principle established in Section 1.2.

IV. THEORY OF PLATES BY VARIATIONAL METHOD

So far, the equations of motion of plates and shells have been obtained by using the method of integration of the three-dimensional equations across the thickness of the undeformed thin body. The effects of eventual simplifications, however, whether for plates or shells, become easier to trace, and a consistent system of equations to derive when a general variational procedure is adopted. For these reasons, we now pass to the derivation of the fundamental equations of the theory of plates in the reference state from the variational principle established in Section 1.2.

A similar derivation for shells is given in the next section.

We shall perform the derivation while employing the linear version (28), of the stress-strain relations. However, the approach leading to (45) can equally well be incorporated into the variational principle (31), if the form of the strain energy function therein is kept arbitrary.

4.1. Evaluation of the Variational Equation

For plates, the various terms in (31) can be evaluated with the help of the relations developed in Section 1.3.

Thus, from (39)

$$s^{ij}\gamma_{ij} = s^{\alpha\beta}\left[{}_0\gamma_{\alpha\beta} + \theta^3\,{}_1\gamma_{\alpha\beta} + (\theta^3)^2\,{}_2\gamma_{\alpha\beta}\right] + \\ + 2\,s^{\alpha 3}\left({}_0\gamma_{\alpha 3} + \theta^3\,{}_1\gamma_{\alpha 3}\right) + s^{33}\,{}_0\gamma_{33} \quad . \tag{80}$$

In accordance with [26], by generalization to the anisotropic case, and using (30)

$$
\begin{aligned}
\Sigma^{*} &= \tfrac{1}{2}\, s^{ij}(\gamma_{ij} - \alpha_{ij}\Theta) \\
&= \tfrac{1}{2} C^{\alpha\beta\delta\nu}[{}_0\gamma_{\alpha\beta}\,{}_0\gamma_{\delta\nu} + \theta^3({}_1\gamma_{\alpha\beta}\,{}_0\gamma_{\delta\nu} + {}_0\gamma_{\alpha\beta}\,{}_1\gamma_{\delta\nu}) + \\
&+ (\theta^3)^2({}_2\gamma_{\alpha\beta}\,{}_0\gamma_{\delta\nu} + {}_1\gamma_{\alpha\beta}\,{}_1\gamma_{\delta\nu} + {}_0\gamma_{\alpha\beta}\,{}_2\gamma_{\delta\nu}) + (\theta^3)^3({}_2\gamma_{\alpha\beta}\,{}_1\gamma_{\delta\nu} + {}_1\gamma_{\alpha\beta}\,{}_2\gamma_{\delta\nu}) + \\
&+ (\theta^3)^4\,{}_2\gamma_{\alpha\beta}\,{}_2\gamma_{\delta\nu}] + C^{\alpha\beta33}\,{}_0\gamma_{33}[{}_0\gamma_{\alpha\beta} + \theta^3\,{}_1\gamma_{\alpha\beta} + (\theta^3)^2\,{}_2\gamma_{\alpha\beta}] + \\
&+ \tfrac{1}{2} C^{3333}({}_0\gamma_{33})^2 + 2C^{\alpha3\beta3}[{}_0\gamma_{\alpha3}\,{}_0\gamma_{\beta3} + \theta^3({}_0\gamma_{\alpha3}\,{}_1\gamma_{\beta3} + {}_1\gamma_{\alpha3}\,{}_0\gamma_{\beta3}) + \\
&+ (\theta^3)^2\,{}_1\gamma_{\alpha3}\,{}_1\gamma_{\beta3}] - \Theta\{C^{\alpha\beta\delta\nu}\alpha_{\alpha\beta}[{}_0\gamma_{\delta\nu} + \theta^3\,{}_1\gamma_{\delta\nu} + (\theta^3)^2\,{}_2\gamma_{\delta\nu}] + \\
&+ C^{\alpha\beta33}\alpha_{\alpha\beta}\,{}_0\gamma_{33} + C^{\alpha\beta33}\alpha_{33}[{}_0\gamma_{\alpha\beta} + \theta^3\,{}_1\gamma_{\alpha\beta} + (\theta^3)^2\,{}_2\gamma_{\alpha\beta}] + \\
&+ 4C^{\alpha3\beta3}\alpha_{\alpha3}({}_0\gamma_{\beta3} + \theta^3\,{}_1\gamma_{\beta3}) + C^{3333}\alpha_{33}\,{}_0\gamma_{33}\} + \\
&+ (\Theta)^2[\tfrac{1}{2} C^{\alpha\beta\delta\nu}\alpha_{\alpha\beta}\alpha_{\delta\nu} + C^{\alpha\beta33}\alpha_{\alpha\beta}\alpha_{33} + \\
&+ 2C^{\alpha3\beta3}\alpha_{\alpha3}\alpha_{\beta3} + \tfrac{1}{2} C^{3333}(\alpha_{33})^2] \quad .
\end{aligned}
\tag{81}
$$

From (38)

$$
\begin{aligned}
&\tfrac{1}{2} s^{ij}(v_i|_j + v_j|_i + v^r|_i\, v_r|_j) \\
&= \tfrac{1}{2} s^{\alpha\beta}[u_\alpha|_\beta + u_\beta|_\alpha + u_\delta|_\alpha u^\delta|_\beta + w_{,\alpha} w_{,\beta} + \\
&+ \theta^3(\psi_\alpha|_\beta + \psi_\beta|_\alpha + u_\delta|_\alpha \psi^\delta|_\beta + \psi_\delta|_\alpha u^\delta|_\beta + w_{,\alpha} w_{1,\beta} + \\
&+ w_{,\beta} w_{1,\alpha}) + (\theta^3)^2(\psi_\delta|_\alpha \psi^\delta|_\beta + w_{1,\alpha} w_{1,\beta})] + \\
&+ s^{\alpha3}[\psi_\alpha + w_{,\alpha} + \psi_\beta u^\beta|_\alpha + w_1 w_{,\alpha} + \\
&+ \theta^3(w_{1,\alpha} + \psi_\beta \psi^\beta|_\alpha + w_1 w_{1,\alpha})] + \\
&+ \tfrac{1}{2} s^{33}[2w_1 + \psi_\alpha \psi^\alpha + (w_1)^2] \quad ,
\end{aligned}
\tag{82}
$$

and

$$
\begin{aligned}
&\rho_0({}_0F^i - {}_0f^i)\delta v_i = \rho_0\{({}_0F^\alpha - {}_0f^\alpha)\delta u_\alpha + \\
&+ \theta^3({}_0F^\alpha - {}_0f^\alpha)\delta\psi_\alpha + ({}_0F^3 - {}_0f^3)\delta w + \\
&+ \theta^3({}_0F^3 - {}_0f^3)\delta w_1\} \quad .
\end{aligned}
\tag{83}
$$

The surface integrals in (31) are evaluated as follows. For that part of the boundary where the stress vector is prescribed, i.e.,

the faces of the plate and part of the edge,

$$\int_{{}_0A_s} \tilde{s}_*^i v_i \, dA = \int_{{}_0\mathcal{a}} [(p^\alpha - \mathcal{F}^\alpha) u_\alpha + (m^\alpha - \mathcal{M}^\alpha)\psi_\alpha + (p - \mathcal{F}) w + (m - \mathcal{M}) w_1] \, dA + \int_{{}_0\mathcal{C}_s} (\tilde{s}^\alpha u_\alpha + \tilde{t}^\alpha \psi_\alpha + \tilde{s} w + \tilde{t} w_1) \, ds \;, \tag{84}$$

where the definitions

$$\tilde{s}^\alpha = \int_{-h}^{+h} \tilde{s}_*^\alpha \, d\theta^3 \;, \qquad \tilde{t}^\alpha = \int_{-h}^{+h} \tilde{s}_*^\alpha \, \theta^3 d\theta^3 \;,$$
$$\tilde{s} = \int_{-h}^{+h} \tilde{s}_*^3 \, d\theta^3 \;, \qquad \tilde{t} = \int_{-h}^{+h} \tilde{s}_*^3 \, \theta^3 d\theta^3 \;, \tag{85}$$

have been used, in addition to those in (40), and ds denotes an element of arc length along ${}_0\mathcal{C}$. Assuming the part where the displacement vector is prescribed to be a portion of the edge of the plate only,

$$\int_{{}_0A_v} s_*^i (v_i - \tilde{v}_i) \, dA = \int_{{}_0\mathcal{C}_v} [s^\alpha (u_\alpha - \tilde{u}_\alpha) + t^\alpha (\psi_\alpha - \tilde{\psi}_\alpha) + s (w - \tilde{w}) + t (w_1 - \tilde{w}_1)] \, ds \;, \tag{86}$$

where the definitions

$$s^\alpha = \int_{-h}^{+h} s_*^\alpha \, d\theta^3 \;, \qquad t^\alpha = \int_{-h}^{+h} s_*^\alpha \, \theta^3 d\theta^3 \;,$$
$$s = \int_{-h}^{+h} s_*^3 \, d\theta^3 \;, \qquad t = \int_{-h}^{+h} s_*^3 \, \theta^3 d\theta^3 \;, \tag{87}$$

have been adopted. The line integrals in (84) and (86) are along the respective portions of ${}_0\mathcal{C}$ where the stress and displacement vectors are prescribed. The evaluation of (87) in terms of the stress and couple resultants and displacement functions leads to

$$\begin{aligned} s^\alpha &= {}_0n_\beta \, (N^{\beta\delta} k_\delta^\alpha + M^{\beta\delta} \psi^\alpha|_\delta + Q^\beta \psi^\alpha) \;, \\ t^\alpha &= {}_0n_\beta \, (M^{\beta\delta} k_\delta^\alpha + K^{\beta\delta} \psi^\alpha|_\delta + T^\beta \psi^\alpha) \;, \\ s &= {}_0n_\beta \, [N^{\beta\alpha} w_{,\alpha} + M^{\beta\alpha} w_{1,\alpha} + Q^\beta (1 + w_1)] \;, \\ t &= {}_0n_\beta \, [M^{\beta\alpha} w_{,\alpha} + K^{\beta\alpha} w_{1,\alpha} + T^\beta (1 + w_1)] \;, \end{aligned} \tag{88}$$

where (87), (33), (40), and (38) have been employed.

Performing the integration with respect to θ^3 in the volume integral part of (31), using the definitions introduced above, by Green's transformation, and a combination of the resulting surface and line integrals, the fundamental equations of the theory of plates in the reference state are obtained, for arbitrary and independent variations of the quantities indicated in the statement of the variational theorem. The results are summarized in the next subsection.

4.2. Fundamental Equations

The strain-displacement relations in terms of the displacement functions introduced in (38) are

$$\begin{aligned}
{}_0\gamma_{\alpha\beta} &= \tfrac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha + u_\delta|_\alpha u^\delta|_\beta + w_{,\alpha} w_{,\beta}\right) , \\
{}_1\gamma_{\alpha\beta} &= \tfrac{1}{2}\left(\psi_\alpha|_\beta + \psi_\beta|_\alpha + u_\delta|_\alpha \psi^\delta|_\beta + \psi^\delta|_\alpha u_\delta|_\beta + \right. \\
&\qquad \left. + w_{,\alpha} w_{1,\beta} + w_{,\beta} w_{1,\alpha}\right) , \\
{}_2\gamma_{\alpha\beta} &= \tfrac{1}{2}\left(\psi_\delta|_\alpha \psi^\delta|_\beta + w_{1,\alpha} w_{1,\beta}\right) , \\
{}_0\gamma_{\alpha 3} &= \tfrac{1}{2}\left(\psi_\alpha + w_{,\alpha} + \psi_\beta u^\beta|_\alpha + w_1 w_{,\alpha}\right) , \\
{}_1\gamma_{\alpha 3} &= \tfrac{1}{2}\left(w_{1,\alpha} + \psi_\beta \psi^\beta|_\alpha + w_1 w_{1,\alpha}\right) , \\
{}_0\gamma_{33} &= \tfrac{1}{2}\left[2 w_1 + \psi_\alpha \psi^\alpha + (w_1)^2\right] ,
\end{aligned} \tag{89}$$

the corresponding strain components being given by (39).

The plate equations of motion are identical to (65)-(68).

The resultant stress-strain relations are

$$\begin{aligned}
N^{\alpha\beta} &= {}_0B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_1B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_2B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} + \\
&\qquad + {}_0B^{\alpha\beta 33}\,{}_0\gamma_{33} - {}_0\Theta^{\alpha\beta} , \\
M^{\alpha\beta} &= {}_1B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_2B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_3B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} + \\
&\qquad + {}_1B^{\alpha\beta 33}\,{}_0\gamma_{33} - {}_1\Theta^{\alpha\beta} , \\
K^{\alpha\beta} &= {}_2B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_3B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_4B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} +
\end{aligned} \tag{90}$$

$$+\,{}_{2}B^{\alpha\beta 33}\,{}_{0}\gamma_{33} - {}_{2}\Theta^{\alpha\beta} \quad ,$$

$$Q^{\alpha} = 2\,({}_{0}B^{\alpha 3\beta 3}\,{}_{0}\gamma_{\beta 3} + {}_{1}B^{\alpha 3\beta 3}\,{}_{1}\gamma_{\beta 3} - {}_{0}\Theta^{\alpha 3}) \quad ,$$

$$T^{\alpha} = 2\,({}_{1}B^{\alpha 3\beta 3}\,{}_{0}\gamma_{\beta 3} + {}_{2}B^{\alpha 3\beta 3}\,{}_{1}\gamma_{\beta 3} - {}_{1}\Theta^{\alpha 3}) \quad ,$$

$$N^{33} = {}_{0}B^{\alpha\beta 33}\,{}_{0}\gamma_{\alpha\beta} + {}_{1}B^{\alpha\beta 33}\,{}_{1}\gamma_{\alpha\beta} + {}_{2}B^{\alpha\beta 33}\,{}_{2}\gamma_{\alpha\beta} +$$

$$+\, B^{3333}\,{}_{0}\gamma_{33} - \Theta^{33} \quad ,$$

where we have defined, by analogy with [10a],

$$\begin{aligned}
{}_{n}B^{\alpha\beta\delta\nu} &= \int_{-h}^{+h} C^{\alpha\beta\delta\nu}\,(\theta^3)^n\,d\theta^3 \\
&\qquad (n = 0,1,2,3,4) \;, \\
{}_{n}B^{\alpha\beta 33} &= \int_{-h}^{+h} C^{\alpha\beta 33}\,(\theta^3)^n\,d\theta^3 \\
&\qquad (n = 0,1,2) \;, \\
{}_{n}B^{\alpha 3\beta 3} &= \int_{-h}^{+h} C^{\alpha 3\beta 3}\,(\theta^3)^n\,d\theta^3 \\
&\qquad (n = 0,1,2) \;, \\
B^{3333} &= \int_{-h}^{+h} C^{3333}\,d\theta^3 \;,
\end{aligned} \tag{91}$$

and, introduced, as thermal stress and couple resultants per unit length of coordinate curves on ${}_{0}a$,

$$\begin{aligned}
{}_{n}\Theta^{\alpha\beta} &= \int_{-h}^{+h} \Theta\,(C^{\alpha\beta\delta\nu}\alpha_{\delta\nu} + C^{\alpha\beta 33}\alpha_{33})\,(\theta^3)^n\,d\theta^3 \\
&\qquad (n = 0,1,2) \;, \\
{}_{n}\Theta^{\alpha 3} &= \int_{-h}^{+h} \Theta\, C^{\alpha 3\beta 3}\alpha_{\beta 3}\,(\theta^3)^n\,d\theta^3 \\
&\qquad (n = 0,1) \;, \\
\Theta^{33} &= \int_{-h}^{+h} \Theta\,(C^{3333}\alpha_{33} + C^{\alpha\beta 33}\alpha_{\alpha\beta})\,d\theta^3 \;.
\end{aligned} \tag{92}$$

On account of (29), obvious symmetry relations hold for the quantities defined in (91) and (92).

Equations (90) coincide with the results of a direct integration of (30) across the thickness of the undeformed plate for a strain distribution given by (39).

Finally, the stress and displacement boundary conditions are respectively, along ${}_{o}\mathcal{C}_{s}$

$$\begin{aligned}
\tilde{s}^{\alpha} = s^{\alpha} &= {}_{o}n_{\beta}\,(N^{\beta\delta}k^{\alpha}_{\delta} + M^{\beta\delta}\psi^{\alpha}|_{\delta} + Q^{\beta}\psi^{\alpha}) \ , \\
\tilde{t}^{\alpha} = t^{\alpha} &= {}_{o}n_{\beta}\,(M^{\beta\delta}k^{\alpha}_{\delta} + K^{\beta\delta}\psi^{\alpha}|_{\delta} + T^{\beta}\psi^{\alpha}) \ , \\
\tilde{s} = s &= {}_{o}n_{\beta}\,[N^{\beta\alpha}w_{,\alpha} + M^{\beta\alpha}w_{1,\alpha} + Q^{\beta}(1+w_{1})] \ , \\
\tilde{t} = t &= {}_{o}n_{\beta}\,[M^{\beta\alpha}w_{,\alpha} + K^{\beta\alpha}w_{1,\alpha} + T^{\beta}(1+w_{1})] \ ,
\end{aligned} \tag{93}$$

and, along ${}_{o}\mathcal{C}_{v}$

$$\tilde{u}_{\alpha} = u_{\alpha} \ , \quad \tilde{\psi}_{\alpha} = \psi_{\alpha} \ , \quad \tilde{w} = w \ , \quad \tilde{w}_{1} = w_{1} \ . \tag{94}$$

Equations (93) coincide with the results of a direct integration of (33) across the thickness of the undeformed plate as given in (88).

V. THEORY OF SHELLS BY VARIATIONAL METHOD

In the present section we employ the variational principle established in Section 1.2. in order to derive the fundamental equations of the theory of shells in the reference state following the procedure adopted in the previous section for plates.

Again, we shall perform this derivation by employing the linear version (28), of the stress-strain relations, keeping in mind that the approach leading to (60) can equally well be incorporated into the variational principle (31), if the form of the strain energy function therein is kept arbitrary.

5.1. Evaluation of the Variational Equation

For shells, the various terms in (31) can be evaluated with the help of the relations developed in Section 1.4.

Equations (80) and (81) are still valid, keeping in mind that ${}_0\gamma_{\alpha\beta}$, ${}_1\gamma_{\alpha\beta}$, ${}_2\gamma_{\alpha\beta}$, ${}_0\gamma_{\alpha 3}$, ${}_1\gamma_{\alpha 3}$, ${}_0\gamma_{33}$ therein are now more complicated functions of the displacement functions introduced in (51).

From (46)-(48), (51), and (53)

$$\frac{1}{2} s^{ij}(v_i|_j + v_j|_i + v^r|_i v_r|_j)$$

$$= \frac{1}{2} s^{\alpha\beta}[u_\alpha\|_\beta + u_\beta\|_\alpha - 2b_{\alpha\beta}w + u^\delta\|_\alpha u_\delta\|_\beta -$$
$$- b^\delta_\beta w\, u_\delta\|_\alpha - b^\delta_\alpha w\, u_\delta\|_\beta + c_{\alpha\beta}(w)^2 + w_{,\alpha} w_{,\beta} +$$
$$+ b^\delta_\beta u_\delta w_{,\alpha} + b^\delta_\alpha u_\delta w_{,\beta} + b^\delta_\beta b^\nu_\alpha u_\delta u_\nu +$$
$$+ \theta^3(\psi_\alpha\|_\beta + \psi_\beta\|_\alpha - 2b_{\alpha\beta}w_1 - b^\delta_\alpha u_\delta\|_\beta - b^\delta_\beta u_\delta\|_\alpha +$$
$$+ 2c_{\alpha\beta}w + u^\delta\|_\alpha \psi_\delta\|_\beta + u^\delta\|_\beta \psi_\delta\|_\alpha - b^\delta_\beta u_\delta\|_\alpha w_1 -$$

$$
\begin{aligned}
& - b^{\delta}_{\alpha} u_{\delta}\|_{\beta} w_1 - b^{\delta}_{\beta} \psi_{\delta}\|_{\alpha} w - b^{\delta}_{\alpha} \psi_{\delta}\|_{\beta} w + 2 c_{\alpha\beta} w w_1 + \\
& + w_{,\alpha} w_{1,\beta} + w_{,\beta} w_{1,\alpha} + b^{\delta}_{\beta} \psi_{\delta} w_{,\alpha} + b^{\delta}_{\alpha} \psi_{\delta} w_{,\beta} + \qquad (95) \\
& + b^{\delta}_{\beta} u_{\delta} w_{1,\alpha} + b^{\delta}_{\alpha} u_{\delta} w_{1,\beta} + b^{\delta}_{\beta} b^{\nu}_{\alpha} u_{\nu} \psi_{\delta} + \\
& + b^{\delta}_{\alpha} b^{\nu}_{\beta} u_{\nu} \psi_{\delta}) + (\theta^3)^2 (- b^{\delta}_{\alpha} \psi_{\delta}\|_{\beta} - b^{\delta}_{\beta} \psi_{\delta}\|_{\alpha} + 2 c_{\alpha\beta} w_1 + \\
& + \psi^{\delta}\|_{\alpha} \psi_{\delta}\|_{\beta} - b^{\delta}_{\beta} w_1 \psi_{\delta}\|_{\alpha} - b^{\delta}_{\alpha} w_1 \psi_{\delta}\|_{\beta} + c_{\alpha\beta} (w_1)^2 + \\
& + w_{1,\alpha} w_{1,\beta} + b^{\delta}_{\beta} \psi_{\delta} w_{1,\alpha} + b^{\delta}_{\alpha} \psi_{\delta} w_{1,\beta} + \\
& + b^{\nu}_{\alpha} b^{\delta}_{\beta} \psi_{\nu} \psi_{\delta})] + s^{\alpha 3} [\psi_{\alpha} + w_{,\alpha} + b^{\beta}_{\alpha} u_{\beta} + \\
& + \psi_{\beta} u^{\beta}\|_{\alpha} - b^{\beta}_{\alpha} \psi_{\beta} w + w_1 w_{,\alpha} + b^{\beta}_{\alpha} u_{\beta} w_1 + \\
& + \theta^3 (w_{1,\alpha} + \psi_{\beta} \psi^{\beta}\|_{\alpha} + w_1 w_{1,\alpha})] + \\
& + \tfrac{1}{2} s^{33} [2 w_1 + \psi^{\alpha} \psi_{\alpha} + (w_1)^2] \; ,
\end{aligned}
$$

and, from (51) and (53)

$$
\begin{aligned}
& \rho_0 ({}_0f^{i} - {}_0F^{i}) \delta v_i = \rho_0 \{ ({}_0f^{\alpha} - {}_0F^{\alpha}) \mu^{\beta}_{\alpha} \delta u_{\beta} + \\
& + \theta^3 ({}_0f^{\alpha} - {}_0F^{\alpha}) \mu^{\beta}_{\alpha} \delta \psi_{\beta} + ({}_0f^{3} - {}_0F^{3}) \delta w + \qquad (96) \\
& + \theta^3 ({}_0f^{3} - {}_0F^{3}) \delta w_1 \} \; .
\end{aligned}
$$

The surface integrals in (31) are evaluated as follows. For that part of the boundary where the stress vector is prescribed, i.e., the faces of the shell and part of the edge, with

$$ {}_0n_{\alpha} \, ds^* = \mu \, {}_0\bar{n}_{\alpha} \, ds \; , \qquad (97) $$

ds^* being an element of arc length along the intersection of the edge boundary and a surface θ^3= const.,

$$
\begin{aligned}
& \int_{{}_0A_s} \tilde{s}^{i}_{*} v_i \, dA = \int_{{}_0a} [(p^{\alpha} - \mathcal{F}^{\alpha}) u_{\alpha} + (m^{\alpha} - \mathcal{M}^{\alpha}) \psi_{\alpha} + \\
& + (p - \mathcal{F}) w + (m - c) w_1] \, dA + \qquad (98) \\
& + \int_{{}_0c_s} (\tilde{s}^{\alpha} u_{\alpha} + \tilde{t}^{\alpha} \psi_{\alpha} + \tilde{s} w + \tilde{t} w_1) \, ds \; ,
\end{aligned}
$$

where (46)-(48), (51), (53), and the definitions

$$
\begin{aligned}
\tilde{s}^{\alpha} &= {}_o\bar{n}_{\delta}\ \widetilde{\int_{-h}^{+h} \mu\mu^{\alpha}_{\beta}\ s^{\delta i}(\delta^{\beta}_{i} + v^{\beta}|_{i})\, d\theta^3}\ , \\
\tilde{t}^{\alpha} &= {}_o\bar{n}_{\delta}\ \widetilde{\int_{-h}^{+h} \mu\mu^{\alpha}_{\beta}\ s^{\delta i}(\delta^{\beta}_{i} + v^{\beta}|_{i})\, \theta^3 d\theta^3}\ , \\
\tilde{s} &= {}_o\bar{n}_{\alpha}\ \widetilde{\int_{-h}^{+h} \mu\, s^{\alpha i}(\delta^{3}_{i} + v^{3}|_{i})\, d\theta^3}\ , \\
\tilde{t} &= {}_o\bar{n}_{\alpha}\ \widetilde{\int_{-h}^{+h} \mu\, s^{\alpha i}(\delta^{3}_{i} + v^{3}|_{i})\, \theta^3 d\theta^3}
\end{aligned}
\qquad (99)
$$

have been used in addition to those in (57). Assuming the part where the displacement vector is prescribed to be a portion of the edge of the shell only,

$$
\begin{aligned}
\int_{{}_oA_v} s^{i}_{*}(v_i - \tilde{v}_i)\, dA &= \int_{{}_o\mathcal{C}_v} [\, s^{\alpha}(u_{\alpha} - \tilde{u}_{\alpha}) + t^{\alpha}(\psi_{\alpha} - \tilde{\psi}_{\alpha}) + \\
&\quad + s^3(w - \tilde{w}) + t^3(w_1 - \tilde{w}_1)\,]\, ds\ ,
\end{aligned}
\qquad (100)
$$

where (51), (53) and the definitions

$$
\begin{aligned}
s^{\alpha} &= {}_o\bar{n}_{\delta} \int_{-h}^{+h} \mu\mu^{\alpha}_{\beta}\, s^{\delta i}(\delta^{\beta}_{i} + v^{\beta}|_{i})\, d\theta^3\ , \\
t^{\alpha} &= {}_o\bar{n}_{\delta} \int_{-h}^{+h} \mu\mu^{\alpha}_{\beta}\, s^{\delta i}(\delta^{\beta}_{i} + v^{\beta}|_{i})\, \theta^3 d\theta^3\ , \\
s &= {}_o\bar{n}_{\alpha} \int_{-h}^{+h} \mu\, s^{\alpha i}(\delta^{3}_{i} + v^{3}|_{i})\, d\theta^3\ , \\
t &= {}_o\bar{n}_{\alpha} \int_{-h}^{+h} \mu\, s^{\alpha i}(\delta^{3}_{i} + v^{3}|_{i})\, \theta^3 d\theta^3\ ,
\end{aligned}
\qquad (101)
$$

have been employed. The line integrals in (98) and (100) are along the respective portions of ${}_o\mathcal{C}$ where the stress and displacement vectors are prescribed.

The evaluation of (101) in terms of the resultant stress and displacement functions leads to

$$
\begin{aligned}
s^{\alpha} &= {}_o\bar{n}_{\beta}\{\, 'N^{\beta\delta}(k^{\alpha}_{\delta} - b^{\alpha}_{\delta} w) + {}'M^{\beta\delta}[\psi^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}(1 + w_1)] + Q^{\beta}\psi^{\alpha}\}\ , \\
t^{\alpha} &= {}_o\bar{n}_{\beta}\{\, 'M^{\beta\delta}(k^{\alpha}_{\delta} - b^{\alpha}_{\delta} w) + {}'K^{\beta\delta}[\psi^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}(1 + w_1)] + T^{\beta}\psi^{\alpha}\}\ , \\
s &= {}_o\bar{n}_{\beta}[\, 'N^{\beta\alpha}(w_{,\alpha} + b^{\delta}_{\alpha} u_{\delta}) + {}'M^{\beta\alpha}(w_{1,\alpha} + b^{\delta}_{\alpha}\psi_{\delta}) + Q^{\beta}(1 + w_1)]\ , \\
t &= {}_o\bar{n}_{\beta}[\, 'M^{\beta\alpha}(w_{,\alpha} + b^{\delta}_{\alpha} u_{\delta}) + {}'K^{\beta\alpha}(w_{1,\alpha} + b^{\delta}_{\alpha}\psi_{\delta}) + T^{\beta}(1 + w_1)]\ ,
\end{aligned}
\qquad (102)
$$

where (101), (33), (57), (46), (51), (53), and (58) have been used.

Performing the integration with respect to θ^3 in the volume integral part of (31) where

$$dV = \mu \, d\theta^3 dA \;, \tag{103}$$

the fundamental equations follow by Green's transformation, a combination of the resulting integrals, and for arbitrary and independent variations of the quantities mentioned in the statement of the variational principle in Section 1.2. The results are summarized in the following subsection.

5.2. Fundamental Equations

The strain-displacement relations in terms of the displacement functions introduced in (51) are

$$\begin{aligned}
{}_0\gamma_{\alpha\beta} &= \tfrac{1}{2}\big[u_\alpha\|_\beta + u_\beta\|_\alpha - 2b_{\alpha\beta}w + u^\delta\|_\alpha u_\delta\|_\beta - \\
&\quad - b^\delta_\beta w u_\delta\|_\alpha - b^\delta_\alpha w u_\delta\|_\beta + c_{\alpha\beta}(w)^2 + \\
&\quad + w_{,\alpha}w_{,\beta} + b^\delta_\beta u_\delta w_{,\alpha} + b^\delta_\alpha u_\delta w_{,\beta} + b^\delta_\beta b^\nu_\alpha u_\delta u_\nu\big] \,, \\
{}_1\gamma_{\alpha\beta} &= \tfrac{1}{2}\big(\psi_\alpha\|_\beta + \psi_\beta\|_\alpha - 2b_{\alpha\beta}w_1 - b^\delta_\alpha u_\delta\|_\beta - b^\delta_\beta u_\delta\|_\alpha + \\
&\quad + 2c_{\alpha\beta}w + u^\delta\|_\alpha \psi_\delta\|_\beta + u^\delta\|_\beta \psi_\delta\|_\alpha - b^\delta_\beta w_1 u_\delta\|_\alpha - \\
&\quad - b^\delta_\alpha w_1 u_\delta\|_\beta - b^\delta_\beta w \psi_\delta\|_\alpha - b^\delta_\alpha w \psi_\delta\|_\beta + 2c_{\alpha\beta}ww_1 + \\
&\quad + w_{,\alpha}w_{1,\beta} + w_{,\beta}w_{1,\alpha} + b^\delta_\beta \psi_\delta w_{,\alpha} + b^\delta_\alpha \psi_\delta w_{,\beta} + \\
&\quad + b^\delta_\beta u_\delta w_{1,\alpha} + b^\delta_\alpha u_\delta w_{1,\beta} + b^\delta_\beta b^\nu_\alpha u_\nu \psi_\delta + b^\delta_\alpha b^\nu_\beta u_\nu \psi_\delta\big) \,, \\
{}_2\gamma_{\alpha\beta} &= \tfrac{1}{2}\big[-b^\delta_\alpha \psi_\delta\|_\beta - b^\delta_\beta \psi_\delta\|_\alpha + 2c_{\alpha\beta}w_1 + \psi_\delta\|_\beta \psi^\delta\|_\alpha - \\
&\quad - b^\delta_\beta \psi_\delta\|_\alpha w_1 - b^\delta_\alpha \psi_\delta\|_\beta w_1 + c_{\alpha\beta}(w_1)^2 + w_{1,\alpha}w_{1,\beta} + \\
&\quad + b^\delta_\beta \psi_\delta w_{1,\alpha} + b^\delta_\alpha \psi_\delta w_{1,\beta} + b^\delta_\beta b^\nu_\alpha \psi_\delta \psi_\nu\big] \,,
\end{aligned} \tag{104}$$

$$
\begin{aligned}
{}_0\gamma_{\alpha 3} &= \tfrac{1}{2}\left(\psi_\alpha + w_{,\alpha} + b^\beta_\alpha u_\beta + \psi_\beta u^\beta\|_\alpha - b^\beta_\alpha \psi_\beta w + \right.\\
&\left. + w_1 w_{,\alpha} + b^\beta_\alpha u_\beta w_1\right) \quad ,\\
{}_1\gamma_{\alpha 3} &= \tfrac{1}{2}\left(w_{1,\alpha} + \psi_\beta \psi^\beta\|_\alpha + w_1 w_{1,\alpha}\right) \quad ,\\
{}_0\gamma_{33} &= \tfrac{1}{2}\left[2 w_1 + \psi_\alpha \psi^\alpha + (w_1)^2\right] \quad ,
\end{aligned}
$$

the corresponding strain components being given by (52).

The shell equations of motion are identical to (74)-(77).

The resultant stress-strain relations are

$$
\begin{aligned}
{}'N^{\alpha\beta} &= {}_0B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_1B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_2B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} + \\
&+ {}_0B^{\alpha\beta 33}\,{}_0\gamma_{33} - {}_0\Theta^{\alpha\beta} \quad ,\\
{}'M^{\alpha\beta} &= {}_1B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_2B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_3B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} + \\
&+ {}_1B^{\alpha\beta 33}\,{}_0\gamma_{33} - {}_1\Theta^{\alpha\beta} \quad ,\\
{}'K^{\alpha\beta} &= {}_2B^{\alpha\beta\delta\nu}\,{}_0\gamma_{\delta\nu} + {}_3B^{\alpha\beta\delta\nu}\,{}_1\gamma_{\delta\nu} + {}_4B^{\alpha\beta\delta\nu}\,{}_2\gamma_{\delta\nu} + \\
&+ {}_2B^{\alpha\beta 33}\,{}_0\gamma_{33} - {}_2\Theta^{\alpha\beta} \quad ,\\
Q^\alpha &= 2\left({}_0B^{\alpha 3\beta 3}\,{}_0\gamma_{\beta 3} + {}_1B^{\alpha 3\beta 3}\,{}_1\gamma_{\beta 3} - {}_0\Theta^{\alpha 3}\right) ,\\
T^\alpha &= 2\left({}_1B^{\alpha 3\beta 3}\,{}_0\gamma_{\beta 3} + {}_2B^{\alpha 3\beta 3}\,{}_1\gamma_{\beta 3} - {}_1\Theta^{\alpha 3}\right) ,\\
N^{33} &= {}_0B^{\alpha\beta 33}\,{}_0\gamma_{\alpha\beta} + {}_1B^{\alpha\beta 33}\,{}_1\gamma_{\alpha\beta} + {}_2B^{\alpha\beta 33}\,{}_2\gamma_{\alpha\beta} + \\
&+ B^{3333}\,{}_0\gamma_{33} - \Theta_{33} \quad ,
\end{aligned}
\tag{105}
$$

where we have defined, by analogy with [10a],

$$
\begin{aligned}
{}_nB^{\alpha\beta\delta\nu} &= \int_{-h}^{+h} \mu\, C^{\alpha\beta\delta\nu} (\theta^3)^n d\theta^3 \\
&\qquad (n = 0, 1, 2, 3, 4) \quad ,\\
{}_nB^{\alpha\beta 33} &= \int_{-h}^{+h} \mu\, C^{\alpha\beta 33} (\theta^3)^n d\theta^3 \\
&\qquad (n = 0, 1, 2) \quad ,\\
{}_nB^{\alpha 3\beta 3} &= \int_{-h}^{+h} \mu\, C^{\alpha 3\beta 3} (\theta^3)^n d\theta^3 \\
&\qquad (n = 0, 1, 2) \quad ,\\
B^{3333} &= \int_{-h}^{+h} \mu\, C^{3333} d\theta^3 \quad ,
\end{aligned}
\tag{106}
$$

and, introduced, as thermal stress and couple resultants per unit length of coordinate curves on ${}_{o}a$,

$$
\begin{aligned}
{}_{n}\Theta^{\alpha\beta} &= \int_{-h}^{+h} \mu\, \Theta \left(C^{\alpha\beta\delta\nu} \alpha_{\delta\nu} + C^{\alpha\beta 33} \alpha_{33} \right) (\theta^3)^n d\theta^3 \\
&\qquad (n = 0, 1, 2), \\
{}_{n}\Theta^{\alpha 3} &= \int_{-h}^{+h} \mu\, \Theta\, C^{\alpha 3 \beta 3} \alpha_{\beta 3} (\theta^3)^n d\theta^3 \\
&\qquad (n = 0, 1), \\
\Theta^{33} &= \int_{-h}^{+h} \mu\, \Theta \left(C^{\alpha\beta 33} \alpha_{\alpha\beta} + C^{3333} \alpha_{33} \right) d\theta^3 .
\end{aligned}
\tag{107}
$$

On account of (29) obvious symmetry relations hold for the quantities defined in (106) and (107). From (58) and (105) follow the stress-strain relations for the unprimed resultants.

Finally, the stress and displacement boundary conditions are respectively, along ${}_{o}\mathcal{C}_s$,

$$
\begin{aligned}
\tilde{s}^{\alpha} = s^{\alpha} &= {}_{o}\bar{n}_{\beta} \{ 'N^{\beta\delta} (k^{\alpha}_{\delta} - b^{\alpha}_{\delta} w) + 'M^{\beta\delta} [\psi^{\alpha} \|_{\delta} - b^{\alpha}_{\delta} (1 + w_1)] + \\
&\quad + Q^{\beta} \psi^{\alpha} \} , \\
\tilde{t}^{\alpha} = t^{\alpha} &= {}_{o}\bar{n}_{\beta} \{ 'M^{\beta\delta} (k^{\alpha}_{\delta} - b^{\alpha}_{\delta} w) + 'K^{\beta\delta} [\psi^{\alpha} \|_{\delta} - b^{\alpha}_{\delta} (1 + w_1)] + \\
&\quad + T^{\beta} \psi^{\alpha} \} , \\
\tilde{s} = s &= {}_{o}\bar{n}_{\beta} ['N^{\beta\alpha} (w_{,\alpha} + b^{\delta}_{\alpha} u_{\delta}) + 'M^{\beta\alpha} (w_{1,\alpha} + b^{\delta}_{\alpha} \psi_{\delta}) + \\
&\quad + Q^{\beta} (1 + w_1)] , \\
\tilde{t} = t &= {}_{o}\bar{n}_{\beta} ['M^{\beta\alpha} (w_{,\alpha} + b^{\delta}_{\alpha} u_{\delta}) + 'K^{\beta\alpha} (w_{1,\alpha} + b^{\delta}_{\alpha} \psi_{\delta}) + \\
&\quad + T^{\beta} (1 + w_1)] ,
\end{aligned}
\tag{108}
$$

and, along ${}_{o}\mathcal{C}_v$,

$$
\tilde{u}_{\alpha} = u_{\alpha} , \quad \tilde{\psi}_{\alpha} = \psi_{\alpha} , \quad \tilde{w} = w , \quad \tilde{w}_1 = w_1 . \tag{109}
$$

Equations (108) coincide with the results of a direct integration of (33) across the thickness of the undeformed shell as given in (102).

VI. CONCLUDING REMARKS

To summarize, two methods have been used in developing the fundamental equations of the theories of plates and shells in terms of a reference state: (a) the integration of the three-dimensional equations across the thickness of the undeformed thin body, (b) the use of a modified version of the Hellinger-Reissner variational theorem of three-dimensional elasticity in terms of a reference state.

Perhaps the most significant feature of the theory is its adoption of the idea of a stress vector measured per unit area of the undeformed body and the related stress tensors that arise as this vector is in turn referred to base vectors of the deformed and the undeformed body.

The two methods have been illustrated for the case when the displacement components, when "shifted" for the case of shells, can be assumed to vary linearly across the thickness of the thin body.

In addition to general stress-strain relations for plates or shells in terms of a strain energy function defined per unit area of the undeformed middle plane or surface, more specific linear stress-strain relations have been employed for an anisotropic material having one plane of elastic symmetry only and including the effect of a prescribed steady temperature field.

While the first method that we have used is as reliable as its age, the second method, when further considerations such as those discussed in [2c], [8b] and references therein are taken into account, has the advantage of producing a complete set of fundamental equations

consistent with various stages of linearization in the general strain-displacement relations. The literature abounds in such intermediate theories, and our work is hoped to have shed some light on these special cases as well as cleared the way for a systematic development of plate and shell theories directly from the three-dimensional theory of elasticity in terms of a reference state.

APPENDIX

NOTATION AND TERMINOLOGY OF SOME EARLIER WRITERS

(a) For s^{ij}:

Novozhilov (1948)	σ^{*}_{ij} $(i = x,y,z; j = x,y,z)$
Truesdell (1952)	T^{KL}
E. Reissner (1953)	s_{ij}, pseudostress
Washizu (1955)	t^{ij}
Doyle-Ericksen (1956)	T^{KL}, Kirchhoff stress
Green-Adkins (1960)	s^{ij}
Herrmann-Armenákas (1960)	σ_{ij}, σ^{*}_{ij}, Trefftz stress
Prager (1961)	$\bar{S}_{ij}$, Kirchhoff stress
Eringen (1962)	T^{KL}, pseudostress
Yu (1964)	σ_{ij}, Kirchhoff-Trefftz stress

(b) For t_{ij}:

Novozhilov (1948)	σ^{*}_{ij} $(i = x,y,z; j = \xi,\eta,\zeta)$
Truesdell (1952)	$T^{K\ell}$
Doyle-Ericksen (1956)	τ^{KL}
Koppe (1956)	$T^{\alpha\beta}$, ersatzspannung
Landau-Lifshitz (1959)	σ_{ij}
C. E. Pearson (1959)	τ°_{ij}, nominal stress
Prager (1961)	$\bar{\bar{T}}_{ij}$, Lagrangian stress
Eringen (1962)	$T^{K\ell}$, pseudostress, Piola stress

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BIOGRAPHICAL SKETCH

The author was born in 1936 in Istanbul, Turkey. He attended the Lycée Saint-Michel in Istanbul and obtained the equivalent of a French Baccalaureate Diploma in the Sciences in 1955. He next was at Robert College Mechanical Engineering Department in Istanbul and graduated in 1959 with a Bachelor of Science degree.

For his graduate studies, the author came to the United States in 1959 and enrolled in the Mechanical Engineering Department of Princeton University, first as a Higgins and then as a Phelps-Dodge Fellow. He received the degree of Master of Science in Engineering in June, 1961.

In September, 1961, the author came to the University of Florida as an Interim Instructor in engineering mechanics. Early in 1962, he enrolled for his doctoral studies in the Department of Engineering Mechanics (now, of Engineering Science and Mechanics) and attended the Institute of Advanced Mechanics held that summer at the Illinois Institute of Technology under the sponsorship of the National Science Foundation. In 1963, the author joined the Advanced Mechanics Research Section as an Assistant in Research, a position he still holds, while continuing his graduate studies. He has since been working with Dr. Ibrahim K. Ebcioglu and Dr. William A. Nash on two research programs, sponsored by the National Science Foundation and the National Aeronautics and Space Administration, respectively, on the theory of sandwich structures in general, and, more specifically, on the elastic behavior of sandwich shells including thermal and geometrically nonlinear effects.

The author is a member of AAUP and a student member of the Society of Engineering Science.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 8, 1964

Thomas L. Martin Jr.

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee

Ibrahim K. Ebcioglu

Chairman

William A. Nash

S. Y. Lu

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